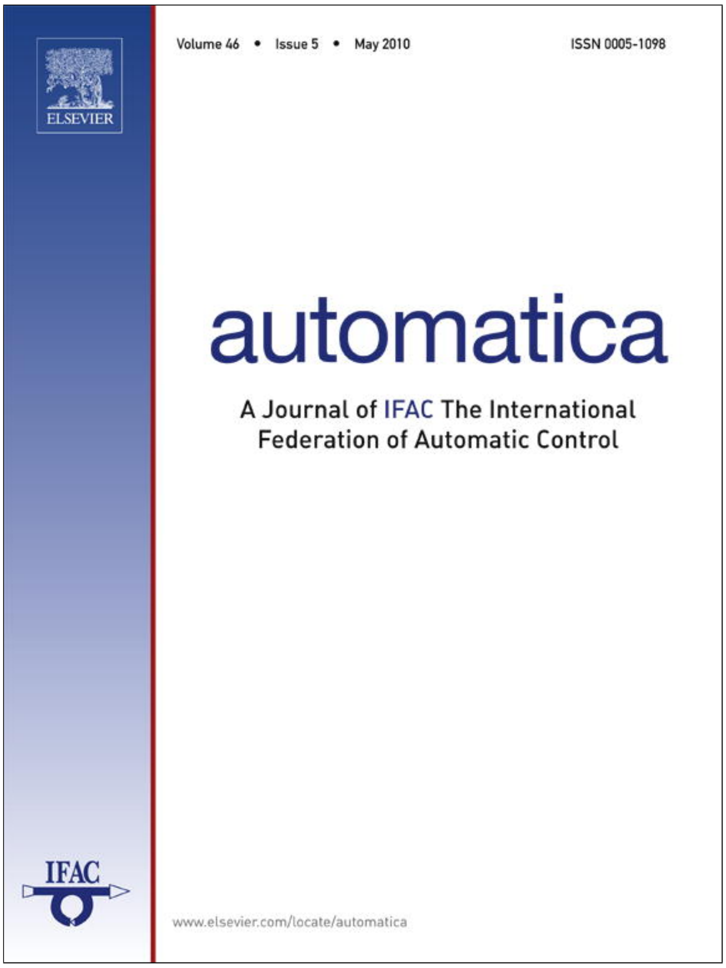
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Brief paper

Stability and reliable data reconstruction of uncertain dynamic systems over finite capacity channels[☆]

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ABSTRACT

In this paper we present an encoder, decoder and a stabilizing controller for reliable data reconstruction and robust stability of uncertain dynamic systems controlled over Additive White Gaussian Noise (AWGN) channels. The uncertainty in the dynamic system is described by a relative entropy constraint. Such an uncertainty description is a natural stochastic generalization of the sum quadratic uncertainty description. This paper complements the results of Farhadi and Charalambous (2008) by showing that the necessary condition presented there can be tight. This is shown by designing an encoder, decoder and a stabilizing controller.

1. Introduction

Recent development in wireless communication and electronics has given birth to micro-electro-mechanical systems (MEMSs) which are small in size and communicate in short distances. These tiny embedded systems, in general, consist of sensors, a data processor, a communication unit and an actuator. They are densely deployed either inside the phenomenon or very close to it. These embedded systems collaborate with each other by exchanging control and observation signals via wireless links. However, due to the limited power of embedded components, the transmission is subject to limited capacity and noise.

In the above applications, the encoders, decoders and controllers must be designed for real-time communication and control when the communication is via limited capacity and noisy communication channels. References Charalambous and Farhadi (2008), Elia (2004), Farhadi and Charalambous (2008), Li and Baillieul (2004), Liberzon and Hespanha (2005), Martins, Dahleh, and Elia (2006), Malyavej and Savkin (2005), Matveev and Savkin (2007), Nair and Evans (2004), Nair, Evans, Mareels, and Moran (2004), Savkin and Petersen (2003), Tatikonda, Sahai, and Mitter (2004) and Yuksel and Basar (2007) are representative although not exhaustive of the recent activity addressing the above questions. They present necessary and sufficient conditions for the stability and reliable data reconstruction of dynamic systems. However, most of these publications are concerned with cases when the dynamic model and communication channel are known. In practice, uncertain dynamic systems and channels are more realistic representations of the actual problems. Only in few publications (e.g., Martins et al., 2006; Matveev & Savkin, 2007) uncertain dynamic systems are considered, in which the uncertain dynamic systems are subject to uniformly bounded disturbances. This excludes dynamic systems which are subject to deterministic or stochastic disturbances of finite energy or power, which are often dealt with using minimax techniques.

This paper addresses control over limited capacity for a class of dynamic systems described by a relative entropy constraint. Such an uncertainty description is a generalization of the sum quadratic uncertainty description considered in Moheimani, Savkin, and Petersen (1995) and Petersen and James (1996). The sum quadratic uncertainty description includes the uniformly bounded uncertainty description as a special case. Consequently, this paper complements the results of Farhadi and Charalambous (2008) by presenting an encoder, a decoder and a controller for uniform reliable data reconstruction and robust stability of an uncertain dynamic system subject to the relative entropy constraint, when it

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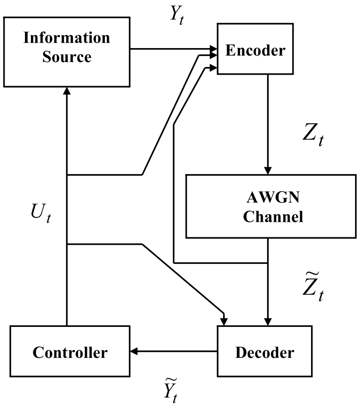

Fig. 1. Control/communication system.

is controlled over AWGN channels. It also complements the results of Farhadi and Charalambous (2008) by calculating the robust entropy rate using stochastic dynamic programming.

The paper is organized as follows. In Section 2, the problem formulation is presented. In Section 3 we summarize the main results of Farhadi and Charalambous (2008) and we calculate the robust entropy rate using stochastic dynamic programming. Then, in Section 4, an encoder, a decoder and a robust controller are presented for uniform reliable data reconstruction and robust stability via AWGN channels. Here, it is shown that the necessary condition presented in Farhadi and Charalambous (2008) is tight. Proofs are given in the Appendix.

2. Problem formulation

Throughout we adopt the following notations. A sequence of random vectors (R.V.s) with length T is denoted by $Y^{T-1} \triangleq (Y_0, Y_1, \ldots, Y_{T-1})$ for $T-1 \in \mathbf{N}_+ \triangleq \{0, 1, 2, \ldots\}$. The density function associated with the R.V. Y is denoted by f_Y. The conditional density function of the R.V. Y given R.V. X is denoted by $f_{Y|X}$. The joint density function of the R.V.s X and Y is denoted by $f_{X,Y}$. The natural logarithm is denoted by $\log(\cdot)$. We denote by $M(q \times o)$ the space of all matrices $A \in \Re^{q\times o}$, and by I_d the identity matrix on the space $M(d \times d)$. We also denote by A' the transpose of A, where A can be either a matrix or a vector, and by $\| \cdot \|_R$ the Euclidean norm with weight R on the finite-dimensional space $\Re^n$. Moreover, we denote by $\mathcal{F}(\mathcal{X})$ the σ-algebra of the subsets of a non-empty (arbitrary) set $\mathcal{X}$; and by $(\mathcal{X}, \mathcal{F}(\mathcal{X}))$ the measurable space. Then, given a pair of measurable spaces $(\bar{A}, \mathcal{F}(\bar{A}))$ and $(\hat{A}, \mathcal{F}(\hat{A}))$, the mapping $Q : \mathcal{F}(\hat{A}) \times \bar{A} \to [0, 1]$ is called a stochastic kernel if it satisfies the following two properties. (i) For every $x \in \bar{A}$, the set function $Q(\cdot|x)$ is a probability measure on $\hat{A}$ and (ii) for every $F \in \mathcal{F}(\hat{A})$, the function $Q(F|\cdot)$ is $\bar{A}$-measurable.

In this paper we are concerned with the control/communication system of Fig. 1, which is defined on a complete probability space $(\Omega, \mathcal{F}(\Omega), P)$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Here, $Y_t \in \Re^d$, $Z_t \in \Re^d$, $\tilde{Z}_t \in \Re^d$, $\tilde{Y}_t \in \Re^d$ and $U_t \in \Re^o$ are random vectors denoting the observations made by the sensors, channel input, channel output, reconstructed version of the observation and the control signal, respectively, at time $t \in \mathbf{N}_+$. This system can be viewed as a basic model for networks of MEMSs and sensor networks. The block diagram of Fig. 1 represents communication between the source of information and the fusion center (where the decoder and controller are located) in a network of MEMSs or sensor networks. In these applications, due to limited communication resources of the source, communication from the source of information to the receiver is subject to communication constraints, as shown in Fig. 1.

The different blocks of Fig. 1 are described below.

Information source. The information source is the output of an uncertain controlled dynamic system with the input $U_t \in \Re^o$, observation (output) $Y_t \in \Re^d$ and the state $X_t \in \Re^n$, where $Y_t = CX_t$ ($C \in M(d \times n)$ is a known matrix). Let $H(\cdot \| \cdot)$ be the relative entropy (Cover & Thomas, 1991), and $f_{Y^{T-1}}$ and $g_{Y^{T-1}}$ be the density function of the observation sequence $Y^{T-1} = (Y_0, Y_1, \ldots, Y_{T-1})$ of the uncertain system and the nominal system (i.e., the controlled dynamic system in the absence of perturbed terms), respectively. Also, let R_c be a known non-negative scalar, and $M = M' \in M(d \times d)$ be a positive semi-definite known matrix. Then, for a given control sequence, the uncertainty in the dynamic system is described by the following relative entropy constraint:

$$f_{Y^{T-1}} \in \mathcal{D}_{SU}(g_{Y^{T-1}}) \triangleq \left\{ f_{Y^{T-1}};\ \frac{1}{T} H(f_{Y^{T-1}} \| g_{Y^{T-1}}) \leq R_c + E\left[\frac{1}{2T}\sum_{t=0}^{T-1} Y_t' M Y_t\right]\right\}, \tag{1}$$

where $E[\cdot]$ denotes the expected value with respect to the probability measure P induced by the sequence Y^{T-1} with the density function $f_{Y^{T-1}}$.

In this paper we are concerned with the following nominal system:

$$\begin{cases} X_{t+1} = AX_t + NU_t + BW_t, \quad X_0 = \xi, \\ Y_t = CX_t, \end{cases} \tag{2}$$

where $X_t \in \Re^n$, $U_t \in \Re^o$, $W_t \in \Re^m$, $Y_t \in \Re^d$, the sequence $\{W_t\}_{t\in\mathbf{N}_+}$ is independent identically distributed (i.i.d.) with distribution $W_t \sim N(0, \Sigma_W)(\Sigma_W > 0)$, and the initial state $\xi \sim N(\bar{x}_0, \bar{V}_0)$ is independent of the sequence $\{W_t\}_{t\in\mathbf{N}_+}$.

The relative entropy $H(f_{Y^{T-1}} \| g_{Y^{T-1}})$ can be thought of as a measure of the difference between the nominal density function $g_{Y^{T-1}}$ induced by (2) and the perturbed (unknown) density function $f_{Y^{T-1}}$. Typical perturbations allowed under the above relative entropy constraint are the perturbations in the mean of the density function $g_{Y^{T-1}}$ (Petersen, James, & Dupuis, 2000). One example of such perturbations is given by the following class of Gauss Markov systems:

$$\begin{cases} X_{t+1} = AX_t + NU_t + BW_t + B\bar{W}_t, \quad X_0 = \xi, \\ Y_t = CX_t, \end{cases} \tag{3}$$

where $X_t \in \Re^n$, $U_t \in \Re^o$, $W_t \in \Re^m$, $\bar{W}_t \in \Re^m$, $Y_t \in \Re^d$, $\{W_t\}_{t\in\mathbf{N}_+}$ is i.i.d. with distribution $W_t \sim N(0, \Sigma_W)(\Sigma_W > 0)$, the initial state $\xi \sim N(\bar{x}_0, \bar{V}_0)$ is independent of $\{W_t\}_{t\in\mathbf{N}_+}$, and $\{\bar{W}_t\}_{t\in\mathbf{N}_+}$ is the perturbed noise random sequence, in which $\bar{W}_t$ is $\{\mathcal{F}_l(W_l); l \leq t-1\}$ adapted and square summable over finite time.

Note that the system described by (2) is the nominal system, while (3) is the uncertain system. Following a similar technique to that used in Petersen et al. (2000), by invoking a change of measure, the chain rule of the relative entropy (Cover & Thomas, 1991, Theorem 2.5.3, p. 23), and the relative entropy formula for two Gaussian density functions, as given in Stroorvogel and Van Shuppen (1994), we can show that the relative entropy of the uncertain system (3) with respect to the nominal system (2) is $H(f_{Y^{T-1}} \| g_{Y^{T-1}}) \triangleq \int \log\left(\frac{f_{Y^{T-1}}(y^{T-1})}{g_{Y^{T-1}}(y^{T-1})}\right) f_{Y^{T-1}}(y^{T-1}) dy^{T-1} = \frac{1}{2}E\left[\sum_{t=0}^{T-2} \bar{W}_t' \Sigma_W^{-1} \bar{W}_t\right]$. That is, the relative entropy constraint (1) holds for the uncertain system (3) with the nominal system (2), provided the following sum quadratic constraint holds.

$$\left\{\{\bar{W}_t\}_{t=0}^{T-2};\ \frac{1}{2T}E\left[\sum_{t=0}^{T-2}\bar{W}_t'\Sigma_W^{-1}\bar{W}_t\right]\right.$$
$$\left.\leq R_c + E\left[\frac{1}{2T}\sum_{t=0}^{T-1}Y_t'MY_t\right]\right\}. \quad (4)$$

Communication channel: We consider an AWGN channel. An AWGN channel is described by the channel input $Z_t \in \Re^d$, channel output $\tilde{Z}_t \in \Re^d$ and the channel noise $\tilde{W}_t \in \Re^d$, which is i.i.d. with distribution $\tilde{W}_t \sim N(0, W_c)$. The channel is described by $\tilde{Z}_t = Z_t + \tilde{W}_t$. This channel at each $t \geq 0$ is subject to the power constraint $E[Z_t'Z_t] \leq P_t$. It is assumed that $\tilde{W}_t$ is independent of the state noise.

Encoder: At each $t \geq 0$, the encoder is described by a stochastic kernel $Q_t^E(C|y^t, u^{t-1}, \tilde{z}^{t-1})$, $C \in \mathcal{F}(\Re^d)$. The encoder uses past and current observation signals, past controls and past channel outputs, and it encodes the current observation to the channel input $Z_t \in \Re^d$.

Decoder: At each $t \geq 0$, the decoder is also described by a stochastic kernel $Q_t^D(C|\tilde{z}^t, u^{t-1})$, $C \in \mathcal{F}(\Re^d)$, which produces the reconstructed version of the observation signal, i.e., $\tilde{Y}_t \in \Re^d$.

Controller: The controller at each $t \geq 0$ is modeled by a stochastic kernel $Q_t^C(C|\tilde{z}^t, u^{t-1})$, $C \in \mathcal{F}(\Re^o)$, which produces the control signal $U_t \in \Re^o$.

In control applications, causality of the encoder and decoder with respect to source messages is required for real-time communication and control. In this paper, we are concerned with reliable data reconstruction of transmitted messages and robust stability as described below.

Definition 2.1 (*Uniform Mean Square Reconstructability*). Consider the control/communication system of Fig. 1, as described above. For a finite $D \geq 0$, the uncertain system is uniformly reconstructed using a mean-square error criterion if there exist a control sequence, a causal encoder, and a causal decoder such that $\lim_{T\to\infty}\frac{1}{T}\sup_{f_{Y^{T-1}}\in\mathcal{D}_{SU}(g_{Y^{T-1}})}\sum_{t=0}^{T-1}E\|Y_t-\tilde{Y}_t\|^2 \leq D$.

Definition 2.2 (*Robust Stability*). Consider the control/communication system of Fig. 1, as described above. For a finite $D^c \geq 0$, the uncertain system is stabilizable if there exist a causal encoder, a causal decoder, and a controller such that $\lim_{T\to\infty}\frac{1}{T}\sup_{f_{Y^{T-1}}\in\mathcal{D}_{SU}(g_{Y^{T-1}})}\sum_{t=0}^{T-1}E\|X_t\|_Q^2 \leq D^c$, for some positive definite matrix $Q = Q' \in M(n\times n)$.

The objective of this paper is to design an encoder, a decoder and a controller for uniform mean-square reconstructability and robust stability, as described above, when the capacity used for transmission is minimum.

3. Robust entropy rate — Necessary condition for reconstructability

As mentioned earlier, this paper particularly complements the results of Farhadi and Charalambous (2008) by presenting an encoder, a decoder and a controller for uniform reliable data reconstruction and robust stability over AWGN channels. Therefore, in the following section, we recall the main results of Farhadi and Charalambous (2008). Moreover, here we identify a connection between H^∞ or robust control techniques and the design of an encoder and a decoder.

Let the directed information from sequence Y^{T-1} to $\tilde{Y}^{T-1}$ be denoted by $I(Y^{T-1}\to\tilde{Y}^{T-1})$, which is defined as follows: $I(Y^{T-1}\to\tilde{Y}^{T-1}) \triangleq \sum_{t=0}^{T-1}I(Y^t;\tilde{Y}_t|\tilde{Y}^{t-1})$ (Massey, 1990), where $I(\cdot;\cdot|\cdot)$ denotes the conditional mutual information (Cover & Thomas, 1991). Also, let the distortion constraint be defined as follows: $\mathcal{D}_D \triangleq \left\{\{f_{\tilde{Y}_t|\tilde{Y}^{t-1},Y^t}\}_{t=0}^{T-1};\ \frac{1}{T}\sum_{t=0}^{T-1}E[\rho(Y_t,\tilde{Y}_t)] \leq D\right\}$, where $\rho(\cdot,\cdot)$ is the distortion measure and let $\mathcal{D}_{SU}^{0,T-1}$ be the class of sources (i.e., $f_{Y^{T-1}} \in \mathcal{D}_{SU}^{0,T-1}$) for a given control sequence. Then, for a given control sequence, the robust sequential rate distortion function for the class of sources $\mathcal{D}_{SU}^{0,T-1}$ is defined by (Farhadi & Charalambous, 2008)

$$R_{SRD,r}^{Y,\tilde{Y}}(D) \triangleq \lim_{T\to\infty}\frac{1}{T}R_{T,r}^{Y,\tilde{Y}}(D),$$
$$R_{T,r}^{Y,\tilde{Y}}(D) \triangleq \inf_{\left\{\{f_{\tilde{Y}_t|\tilde{Y}^{t-1},Y^t}\}_{t=0}^{T-1}\in\mathcal{D}_D\right\}}\ \sup_{f_{Y^{T-1}}\in\mathcal{D}_{SU}^{0,T-1}} I(Y^{T-1}\to\tilde{Y}^{T-1}). \quad (5)$$

Moreover, for a given control sequence, the robust entropy rate is defined by (Farhadi & Charalambous, 2008)

$$\mathcal{H}_r(\mathcal{Y}) \triangleq \lim_{T\to\infty}\frac{1}{T}\sup_{f_{Y^{T-1}}\in\mathcal{D}_{SU}^{0,T-1}} H_S(f_{Y^{T-1}}), \quad (6)$$

where $H_S(f_{Y^{T-1}})$ is the Shannon (differential) entropy, which is defined as follows (Cover & Thomas, 1991): $H_S(f_{Y^{T-1}}) \triangleq -\int\log\left(f_{Y^{T-1}}(y^{T-1})\right)f_{Y^{T-1}}(y^{T-1})dy^{T-1}$. Then, when $\rho(\cdot,\cdot) = \|\cdot\|^r$, $r > 0$, we have the following necessary condition for uniform r-moment reconstructability (i.e., $\lim_{T\to\infty}\frac{1}{T}\sup_{f_{Y^{T-1}}\in\mathcal{D}_{SU}^{0,T-1}}\sum_{t=0}^{T-1}E\|Y_t-\tilde{Y}_t\|^r \leq D$) for the control/communication system of Fig. 1.

Theorem 3.1 (*Farhadi & Charalambous, 2008, Theorem 4.5 together with Theorem 4.7*). *Consider the block diagram of Fig. 1 subject to the conditional independence assumption and uncertainty in the source. Suppose this system is described by discrete memoryless channels (DMCs) (Cover & Thomas, 1991) or an AWGN channel. Then, for a given control sequence, a necessary condition for uniform r-moment reconstructability (as defined above), up to the distortion level D, when the encoder and decoder are causal operations of the source messages, is*

$$\mathcal{C} \geq R_{SRD,r}^{Y,\tilde{Y}}(D) \geq \mathcal{H}_r(\mathcal{Y}) - \frac{d}{2} + \log\left(\frac{2}{dV_d\Gamma(\frac{d}{2})}\left(\frac{d}{2D}\right)^{\frac{d}{2}}\right), \quad (7)$$

where $\mathcal{C}$ denotes the channel capacity, $\Gamma(\cdot)$ denotes the gamma function, and V_d is the volume of the unit sphere (i.e., $V_d = Vol(S_d)$; $S_d \triangleq \{v \in \Re^d;\ \|v\| \leq 1\}$).

Note that the necessary condition (7) is independent of the information available at the encoder, decoder, and controller. The robust sequential rate distortion is very difficult to compute, particularly, for $r \neq 2$. Therefore, the lower bound (7), which involves the entropy rate, is very practical because the entropy rate can be computed much more easily. To illustrate this point, we apply the necessary condition (7) to the uncontrolled version of the uncertain system (3) (i.e., (3) with $U_t = 0$) subject to the sum quadratic constraint (4). This is done by computing the robust entropy rate $\mathcal{H}_r(\mathcal{Y})$ for this system.

Toward this goal, consider the robust entropy rate (6) described by the class $\mathcal{D}_{SU}^{0,T-1}$ as given below:

$$\mathcal{D}_{SU}^{0,T-1} = \mathcal{D}_{SU}(g_{Y^{T-1}}) \triangleq \left\{f_{Y^{T-1}};\ \frac{1}{T}H(f_{Y^{T-1}}\ \|\ g_{Y^{T-1}})\right.$$
$$\left.\leq R_c + E\left[\frac{1}{2T}\sum_{t=0}^{T-1}Y_t'MY_t\right]\right\}, \quad (8)$$

where $f_{Y^{T-1}} \in \mathcal{D}_{SU}(g_{Y^{T-1}})$ corresponds to the uncertain system (3) and $g_{Y^{T-1}}$ corresponds to the nominal system (2) with $U_t = 0$. Thus,

$$\mathcal{H}_r(\mathcal{Y}) = \lim_{T\to\infty} \frac{1}{T} \sup_{f_{Y^{T-1}} \in \mathcal{D}_{SU}(g_{Y^{T-1}})} H_S(f_{Y^{T-1}}). \tag{9}$$

In Farhadi and Charalambous (2008), for the relative entropy constraint (1) with $M = 0$, an explicit solution for the robust entropy rate (9) has been found using calculus of variation. Here, for the uncertain system (3) subject to the sum quadratic constraint (4), an explicit solution for the robust entropy rate is obtained using stochastic dynamic programming (Caines, 1988, pp. 670–677). This result is given in the following theorem. For simplicity here we assume that the matrix C is an identity matrix (i.e., $n = d$ and $C = I_d$).

Theorem 3.2. *Consider the uncontrolled version of the uncertain system (3) (with $C = I_d$) subject to the sum quadratic constraint (4) with $M > 0$. Suppose, for some $s > 0$, we have $(1+s)\Sigma_W^{-1} > B'\Xi_t B$ for all $t \in \{0, 1, 2, \ldots, T-1\}$, where $\Xi_t = A'\Xi_{t+1}A - A'\Xi_{t+1}B[-(1+s)\Sigma_W^{-1} + B'\Xi_{t+1}B]^{-1}B'\Xi_{t+1}A + sM$, $\Xi_{T-1} = sM$. Also, suppose Ξ_∞ is the stabilizing solution of the following Algebraic Riccati equation appearing in the H^∞ estimation and control:*

$$\Xi_\infty = A'\Xi_\infty A - A'\Xi_\infty B[-(1+s)\Sigma_W^{-1} + B'\Xi_\infty B]^{-1} \times B'\Xi_\infty A + sM. \tag{10}$$

That is, for this Ξ_∞, the matrix $A_\Xi \triangleq A' - K_\Xi B'$ is stable, where $K_\Xi \triangleq A'\Xi_\infty B(B'\Xi_\infty B - (1+s)\Sigma_W^{-1})^{-1}$.

Then, if the matrices $(I_d - \Xi_\infty Q_t)^{-1}$ are uniformly bounded for all $t \in \mathbf{N}_+$, where Q_t satisfies the following recursion: $Q_{t+1} = A'_\Xi Q_t A_\Xi + B(B'\Xi_\infty B - (1+s)\Sigma_W^{-1})^{-1}B'$, $Q_{-1} = 0$, we have the convergence of Ξ_t to Ξ_∞, the unique stabilizing solution of Eq. (10), as $T \to \infty$ (Hassibi, Sayed, & Kailath, 1999, Theorem 14.4.1, p. 423). Subsequently, the robust entropy rate (9) for this uncertain system is given by

$$\mathcal{H}_r(\mathcal{Y}) = \frac{d}{2}\log(2\pi e) + \frac{1}{2}\log\det(B\Sigma_W B') + \min_{s>0}\left\{sR_c + \frac{1}{2}trac(B'\Xi_\infty B\Sigma_W)\right\}. \tag{11}$$

Proof. See the Appendix. □

Remark 3.3. In (11), in order to minimize the term $sR_c + \frac{1}{2}trac(B'\Xi_\infty B\Sigma_W)$ over $s > 0$, we need to find the stabilizing solution of the Algebraic Riccati equation (10). This is similar to the problem associated with H^∞ Riccati equations, in which numerical techniques are involved to obtain the optimal s^*. Alternatively, since this is a Lagrange multiplier, one has to use the constraint which holds with equality to find the optimal s^*.

From the results of Theorems 3.1 and 3.2 we have the following corollary for reconstructability of the uncontrolled version of the uncertain system (3) over noisy channels.

Corollary 3.4. *Under the assumptions of Theorems 3.1 and 3.2, a necessary condition for uniform reliable data reconstruction of the observation sequence associated with the uncontrolled version of the uncertain system (3) (with $C = I_d$) subject to the sum quadratic constraint (4) is given by the condition (7) with the robust entropy rate $\mathcal{H}_r(\mathcal{Y})$, as given by (11).*

Note that Corollary 3.4 connects H^∞ or robust control techniques to reconstructability over noisy channels.

4. Encoder, decoder and stabilizing controller

In this section we present an encoder, a decoder and a robust controller for uniform mean-square reconstructability and robust stability, as described by Definitions 2.1 and 2.2.

Consider the control/communication system of Fig. 1, where the information source is the observation sequence of the uncertain controlled dynamic system described by the relative entropy constraint (1) with the nominal system (2). For this system, the encoder and decoder are a linear encoder and a linear decoder, as described below.

Linear encoder: The encoder consists of a pre-encoding scheme that produces $K_t = Y_t - C\hat{X}_t$, where $Y_t \in \Re^d$ is the observation of the uncertain system and $\hat{X}_t \in \Re^n$ is the estimate of the state variable $X_t \in \Re^n$ given sequences $\tilde{Z}^{t-1}$ and U^{t-1}. The encoder multiplies the message K_t by the encoding gain $\alpha_t \in M(d \times d)$, and produces the channel input $Z_t = \alpha_t K_t \in \Re^d$.

Linear decoder: The decoder multiplies the channel outputs by the decoding gain $\gamma_t \in M(d \times d)$, and produces $\tilde{K}_t = \gamma_t \tilde{Z}_t \in \Re^d$. R.V. $\tilde{K}_t$ can be viewed as the reconstructed version of the message $K_t \in \Re^d$. The decoder then produces $\tilde{Y}_t \triangleq \tilde{K}_t + C\hat{X}_t \in \Re^d$, which is the reconstructed version of the observation sequence at the decoder output.

Note that the encoder and decoder, as described above, are causal functions of the source messages.

For a given control sequence, the encoding and decoding gains α_t and γ_t are obtained by generalizing the source-channel matching principle, as described in Gastper, Rimoldi, and Vetterli (2003), to a class of dynamic systems. On the other hand, the controller is obtained using minimax techniques, similar to Petersen et al. (2000), although as it will become clear shortly that the model considered here is quite different because of the structure of the encoder that employs feedback.

Control law and the state estimate: R.V. $\tilde{K}_t \in \Re^d$ can be viewed as the observation vector of a partially observed uncertain dynamic system with the state variable $X_t \in \Re^n$, in which this system is described by the following relative entropy constraint:

$$f_{X^{T-1},\tilde{K}^{T-1}} \in \mathcal{D}_{SU}(g_{X^{T-1},\tilde{K}^{T-1}}) \triangleq \left\{ f_{X^{T-1},\tilde{K}^{T-1}};\ \frac{1}{T}H(f_{X^{T-1},\tilde{K}^{T-1}} \parallel g_{X^{T-1},\tilde{K}^{T-1}}) \le R_c + E\left[\frac{1}{2T}\sum_{t=0}^{T-1} Y_t'MY_t\right], Y_t = CX_t \right\}, \tag{12}$$

where $f_{X^{T-1},\tilde{K}^{T-1}}$ is the perturbed density function corresponding to the uncertain system and $g_{X^{T-1},\tilde{K}^{T-1}}$ is the nominal (known) density function corresponding to the following system:

$$\begin{cases} X_{t+1} = AX_t + NU_t + BW_t, & X_0 = \xi, \\ \tilde{K}_t = \gamma_t\alpha_t(Y_t - C\hat{X}_t) + \gamma_t\tilde{W}_t, & Y_t = CX_t. \end{cases} \tag{13}$$

Consider the following cost functional:

$$\frac{1}{2T}\sum_{t=0}^{T-1} E[\|X_t\|^2_{C'C} + \|U_t\|^2_R],$$

where the weighting matrix $R = R' \in M(o \times o)$ is a given positive definite matrix and $U_t \in \mathcal{G}^U_{t-1} \triangleq \mathcal{F}(\tilde{Z}^{t-1}, U^{t-1})$. The objective is to find the control policy U_t that minimizes the maximum of the above cost functional over the class (12). That is,

(Minimax Problem):

$$J = \lim_{T\to\infty} \inf_{U^{T-1}} \sup_{\{f_{X^{T-1},\tilde{K}^{T-1}} \in \mathcal{D}_{SU}(g_{X^{T-1},\tilde{K}^{T-1}})\}} \frac{1}{2T} \times \sum_{t=0}^{T-1} E[\|X_t\|^2_{C'C} + \|U_t\|^2_R]. \tag{14}$$

Suppose α_t and γ_t are invertible and $\lim_{t\to\infty}\alpha_t = \alpha_\infty$ and $\lim_{t\to\infty}\gamma_t = \gamma_\infty$ exist. The existence of the limits is justified later. For simplicity, without loss of generality, assume there exists a positive definite symmetric matrix $\tilde{\Sigma}_W \in M(o\times o)$ such that $B\Sigma_W B' = N\tilde{\Sigma}_W N'$. Then, following a similar methodology to that used in Petersen et al. (2000), by implementing the Legendre–Fenchel transformation (Pra, Meneghini, & Runggaldier, 1996), we can convert the minimax problem to an equivalent partial information, risk-sensitive optimal control problem. Subsequently, the optimal controller is given by the following equations (Collings, James, & Moore, 1996; Petersen et al., 2000; Whittle, 1981):

$$U_t = K_t\hat{X}_t,$$

$$K_t \triangleq -R^{-1}N'\left(\Pi_{t+1}^{-1} + NR^{-1}N' - \frac{N\tilde{\Sigma}_W N'}{\tau}\right)^{-1} \times A\left(I_n - \frac{\Sigma_t\Pi_t}{\tau}\right)^{-1} \quad (15)$$

$$\hat{X}_{t+1} = A\hat{X}_t + NU_t + T_t\tilde{K}_t + A\left(\Sigma_t^{-1} + C'\alpha_t' W_c^{-1}\alpha_t C - \frac{C'C}{\tau} - C'MC\right)^{-1}\left(\frac{C'C}{\tau} + C'MC\right)\hat{X}_t, \quad \hat{X}_0 = \bar{x}_0, \quad (16)$$

$$T_t = A\left(\Sigma_t^{-1} + C'\alpha_t' W_c^{-1}\alpha_t C - \frac{C'C}{\tau} - C'MC\right)^{-1} \times C'\alpha_t' W_c^{-1}(\gamma_t)^{-1}, \quad (17)$$

with the symmetric matrices Σ_t and Π_t being the solutions of the following indefinite Riccati equations:

$$\Sigma_{t+1} = A\Sigma_t A' - A\Sigma_t C'\left[C\Sigma_t C' + \left(\alpha_t' W_c^{-1}\alpha_t - \frac{I_d}{\tau} - M\right)^{-1}\right]^{-1} \times C\Sigma_t A' + B\Sigma_W B', \qquad \Sigma_0 = \bar{V}_0, \quad (18)$$

$$\Pi_t = A'\Pi_{t+1}A - A'\Pi_{t+1}N\left(N'\Pi_{t+1}N + \left(R^{-1} - \frac{\tilde{\Sigma}_W}{\tau}\right)^{-1}\right)^{-1} \times N'\Pi_{t+1}A + C'C + \tau C'MC, \qquad \Pi_T = 0. \quad (19)$$

Note that the solutions of the above Riccati equations are required to satisfy the following conditions for each $t \in \mathbf{N}_+$:

$$\Sigma_t > 0, \qquad \Sigma_t^{-1} + C'\alpha_t' W_c^{-1}\alpha_t C - \frac{C'C}{\tau} - C'MC > 0,$$
$$\Pi_{t+1}^{-1} - \frac{B\Sigma_W B'}{\tau} > 0, \qquad \Pi_t^{-1} - \frac{\Sigma_t}{\tau} > 0. \quad (20)$$

Since we are interested in the stationary control law, we must introduce assumptions that guarantee the existence of the limits, $\lim_{t\to\infty}\Sigma_t = \Sigma_\infty$ and $\lim_{t\to\infty}\Pi_t = \Pi_\infty$. In the following, we present sufficient conditions for the existence of these limits.

Convergence of the Riccati equations: Let Σ_∞ be the stabilizing solution of the Algebraic Riccati equation corresponding to the Riccati equation (18) (i.e., for this Σ_∞ the matrix $A_\Sigma \triangleq A - K_\Sigma C$ is stable, where $K_\Sigma \triangleq A\Sigma_\infty C'(C\Sigma_\infty C' + (\alpha_\infty' W_c^{-1}\alpha_\infty - \frac{I_d}{\tau} - M)^{-1})^{-1}$). Then, if $\Sigma_0 = \bar{V}_0$ is such that the matrices $(I_n + (\Sigma_0 - \Sigma_\infty)O_t)^{-1}$ are uniformly bounded for all $t \in \mathbf{N}_+$, where O_t satisfies the following recursion: $O_{t+1} = A_\Sigma' O_t A_\Sigma + C'(C\Sigma_\infty C' + (\alpha_\infty' W_c^{-1}\alpha_\infty - \frac{I_d}{\tau} - M)^{-1})^{-1}C$, $O_{-1} = 0$, then Σ_t converges to Σ_∞, the unique stabilizing solution of the Algebraic Riccati equation corresponding to the Riccati equation (18) (Hassibi et al., 1999, Theorem 14.4.1, p. 423). Similarly, let Π_∞ be the stabilizing solution of the Algebraic Riccati equation associated with the Riccati equation (19) (i.e., $A_\Pi \triangleq A' - K_\Pi N'$ is stable, where $K_\Pi \triangleq A'\Pi_\infty N(N'\Pi_\infty N + (R^{-1} - \frac{\tilde{\Sigma}_W}{\tau})^{-1})^{-1}$). Then, if the matrices $(I_n - \Pi_\infty S_t)^{-1}$ are uniformly bounded for all $t \in \mathbf{N}_+$, where S_t satisfies the following recursion: $S_{t+1} = A_\Pi' S_t A_\Pi + N(N'\Pi_\infty N + (R^{-1} - \frac{\tilde{\Sigma}_W}{\tau})^{-1})^{-1}N'$, $S_{-1} = 0$, then Π_t converges to Π_∞, the unique stabilizing solution of the Algebraic Riccati equation associated with the Riccati equation (19).

In the above expressions, $\tau > 0$ is the Lagrange multiplier and is chosen to minimize the value of the cost function, which is given by

$$J = \tau\left(\lim_{T\to\infty}\frac{\tilde{V}_\tau}{T} + R_c\right),$$

$$\lim_{T\to\infty}\frac{\tilde{V}_\tau}{T} = -\frac{1}{2}\log\left(\det\left(I_n - \left(\frac{C'C}{\tau} + C'MC\right)\Sigma_\infty\right)\right) - \frac{1}{2}\log\det\Theta_\infty, \quad (21)$$

where $\Theta_\infty = I_n - \frac{1}{\tau}T_\infty(\gamma_\infty W_c\gamma_\infty' + (\gamma_\infty\alpha_\infty C)(\Sigma_\infty^{-1} - \frac{C'C}{\tau} - C'MC)^{-1}(\gamma_\infty\alpha_\infty C)')T_\infty'(\Pi_\infty^{-1} - \frac{\Sigma_\infty}{\tau})^{-1}$, $T_\infty \triangleq \lim_{t\to\infty} T_t$.

Remark 4.1. (i) The case without uncertainty (i.e., $R_c = 0$ and $M = 0$) corresponds to the case where $\tau \to \infty$. For this case, the results given in (15)–(19) are reduced to the standard LQG results, and α_t and γ_t are given in Charalambous and Farhadi (2008).
(ii) The controller is a certainty equivalent controller in the sense that the estimator and the control law are designed separately and combined via $U_t = K_t\hat{X}_t$.
(iii) For a given weighting matrix $R \in M(o\times o)$, using the certainty equivalent controller (15), we can stabilize all the subsystems of the uncertain system in the following sense: $\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} E\|X_t\|_{C'C}^2 \le 2J = D^c$.

Encoding and decoding gains α_t and γ_t: For simplicity of analysis, from now on we consider the case of $Y_t \in \Re$, and subsequently, the case of $Z_t \in \Re$, $\tilde{Z}_t \in \Re$, and $\tilde{Y}_t \in \Re$. The vector case is treated similarly. The encoding and decoding gains α_t and γ_t are designed for the sequence $\{K_t\}_{t\in\mathbf{N}_+}$ that corresponds to the subsystem (of the uncertain system) which maximizes the payoff functional (14). Note that this sequence carries the maximum entropy (i.e., maximum uncertainty).

Throughout this section it is assumed that the Lagrange multiplier τ is sufficiently large. For sufficiently large enough τ, the sequence $\{K_t\}_{t\in\mathbf{N}_+}$ which corresponds to the subsystem with maximum entropy can be considered as an orthogonal Gaussian sequence with distribution $K_t \sim N(0, \Psi_t)$, where $\Psi_t = C\Sigma_t C'$ and Σ_t is the solution of the following Riccati equation:

$$\Sigma_{t+1} = A\Sigma_t A' - A\Sigma_t C'\left(C\Sigma_t C' + \left(\frac{\alpha_t^2}{W_c} - \frac{1}{\tau} - M\right)^{-1}\right)^{-1} \times C\Sigma_t A' + B\Sigma_W B', \quad \Sigma_0 = \bar{V}_0. \quad (22)$$

Note that in the above equation W_c is the variance of the channel noise $\tilde{W}_t \sim N(0, W_c)$. Also, note that, since the sequence $\{K_t\}_{t\in\mathbf{N}_+}$ ($K_t \sim N(0, \Psi_t)$), as described above, carries the maximum entropy, for each $t \in \mathbf{N}_+$, we have $E[K_t^2] \le \Psi_t$, for all subsystems of the uncertain system.

Next, for a given control sequence and the class $f_{K^{T-1}} \in \mathcal{D}_{SU}^{0,T-1} = \{f_{K^{T-1}}; K_t = Y_t - C\hat{X}_t, E[K_t^2] \le \Psi_t, 0 \le t \le T-1\}$, consider the following maximum rate distortion problem.

$$R^{K,\tilde{K}}_{T,MRD}(D) \triangleq \sup_{f_{K^{T-1}}\in\mathcal{D}^{0,T-1}_{SU}} R^{K,\tilde{K}}_{T}(D),$$
$$R^{K,\tilde{K}}_{T}(D) \triangleq \inf_{\left\{\{f_{\tilde{K}_t|\tilde{K}^{t-1},K^t}\}_{t=0}^{T-1};\ \frac{1}{T}\sum_{t=0}^{T-1}E\|K_t-\tilde{K}_t\|^2\le D\right\}} I(K^{T-1}\to\tilde{K}^{T-1}). \tag{23}$$

From Sakrison (1969), it follows that the maximum of the rate distortion function $R^{K,\tilde{K}}_{T}(D)$ for a class of sources corresponds to a Gaussian source with independent outcomes. Hence, as we have $E[K_t^2] \le \Psi_t$ ($\forall t \in \mathbf{N}_+$) for all subsystems, the maximum of the rate distortion function occurs for the sequence corresponding to the subsystem with maximum entropy. Therefore, the minimizing kernel of the above maximum rate distortion problem is $f^*_{\tilde{K}^{T-1}|K^{T-1}} = \prod_{t=0}^{T-1} f^*_{\tilde{K}_t|K_t}$, where $f^*_{\tilde{K}_t|K_t} \sim N(\eta_t K_t, \eta_t D)$, $\eta_t = 1 - \frac{D}{\Psi_t}$, for $D < \min_{t\in\mathbf{N}_+}\Psi_t$. On the other hand, the conditional density function of the reconstructed sequence $\tilde{K}^{T-1}$ given the transmitted sequence K^{T-1} via the AWGN channel is given by $f_{\tilde{K}^{T-1}|K^{T-1}} = \prod_{t=0}^{T-1} f_{\tilde{K}_t|K_t}, f_{\tilde{K}_t|K_t} \sim N(\gamma_t\alpha_t K_t, \gamma_t W_c)$. The source-channel matching principle, as described in Gastper et al. (2003), requires that $f^*_{\tilde{K}_t|K_t} = f_{\tilde{K}_t|K_t}$, which results in α_t and γ_t as given by

$$\alpha_t = \sqrt{\frac{\eta_t W_c}{D}}, \qquad \gamma_t = \sqrt{\frac{D\eta_t}{W_c}}. \tag{24}$$

Note that under the conditions presented earlier, the limit, $\lim_{t\to\infty}\Sigma_t = \Sigma_\infty$ exists, and therefore the limits $\lim_{t\to\infty}\Psi_t = \Psi_\infty \triangleq C\Sigma_\infty C'$, $\lim_{t\to\infty}\alpha_t = \alpha_\infty$ and $\lim_{t\to\infty}\gamma_t = \gamma_\infty$ also exist. A consequence of choosing α_t and γ_t, as given above, is reliable data reconstruction with minimum required capacity. This result is shown next.

Corollary 4.2. *Consider the linear encoder and decoder, as described earlier, and let α_t and γ_t be given by (24).*
Then

(i) *For each $t \ge 0$, $E\|K_t - \tilde{K}_t\|^2 \le D$ and $E\|Y_t - \tilde{Y}_t\|^2 \le D$, for all subsystems of the uncertain system.*
(ii) *Suppose the power constraint is described by $E[Z_t^2] \le P_t = \sup_{E[K_t^2]} E[(\alpha_t K_t)^2] = \frac{\eta_t W_c}{D}\Psi_t$. Then, uniform mean-square reconstructability is obtained by transmission with minimum required capacity $\mathcal{C} = R^{K,\tilde{K}}_{SRD,r}(D) = \mathcal{H}_r(\mathcal{K}) - \frac{1}{2}\log(2\pi e D)$.*

Proof. See the Appendix. □

Remark 4.3. We have the following general remarks regarding the results of this section.
(i) As discussed in Petersen et al. (2000), using Legendre–Fenchel transformation, the minimax problem (14) is equivalent to the risk-sensitive linear quadratic Gaussian control problem. As shown in Bansal and Basar (1989), in controlling linear Gaussian systems over AWGN channels, the linear encoder and linear decoder, as used in this paper, correspond to the optimal solution.
(ii) It is more reasonable to use the minimizing kernel associated with the robust sequential rate distortion function (5) in the source-channel matching principle. Nevertheless, by implementing the minimizing kernel, which is the solution of the maximin rate distortion function (23), we can show that the necessary condition (7) is tight for uniform reconstructability. Therefore, we considered the maximum rate distortion function, which is easier to work with.

5. Conclusion

This paper complements the results of Farhadi and Charalambous (2008) by showing that the necessary condition presented there can be tight. It also complements Matveev and Savkin (2007) and Martins et al. (2006) by considering a relative entropy uncertainty description. In this paper a connection between the existence of an encoder and a decoder (for reliable data reconstruction) and the H^∞ problem was also established. The encoding, decoding and stabilizing schemes presented in this paper can be used to address uniform reconstructability and robust stability via AWGN channels.

Appendix

Proof of Theorem 3.2. Consider the robust entropy rate (9) described by the relative entropy constraint (8). In general, we have the following relation for the entropy:

$$\begin{aligned} H_S(f_{Y^{T-1}}) &= -\int \log\left(\frac{f_{Y^{T-1}}(y^{T-1})}{g_{Y^{T-1}}(y^{T-1})}\right) f_{Y^{T-1}}(y^{T-1})dy^{T-1} \\ &\quad - \int \log\left(g_{Y^{T-1}}(y^{T-1})\right) f_{Y^{T-1}}(y^{T-1})dy^{T-1} \\ &= -H(f_{Y^{T-1}} \parallel g_{Y^{T-1}}) \\ &\quad - \int \log\left(g_{Y^{T-1}}(y^{T-1})\right) f_{Y^{T-1}}(y^{T-1})dy^{T-1}. \end{aligned} \tag{25}$$

Note that, for the system considered in Theorem 3.2, we have $H(f_{Y^{T-1}} \parallel g_{Y^{T-1}}) = \frac{1}{2}E[\sum_{t=0}^{T-2}\bar{W}_t'\Sigma_W^{-1}\bar{W}_t]$. From the chain rule for density functions, it follows that the nominal density function $g_{Y^{T-1}}$ can be written as follows: $g_{Y^{T-1}} = g_{Y_0}.g_{Y_1|Y_0}.g_{Y_2|Y_0,Y_1}.\cdots.g_{Y_{T-1}|Y^{T-2}}$. Since $Y_0 \to Y_1 \to \cdots \to Y_{T-1}$ forms a Markov chain, $g_{Y^{T-1}} = g_{Y_0}.g_{Y_1|Y_0}.g_{Y_2|Y_1}.\cdots.g_{Y_{T-1}|Y_{T-2}}$. Thus,

$$\begin{aligned} g_{Y^{T-1}} &= \frac{1}{(2\pi)^{\frac{d}{2}}(\det(\bar{V}_0))^{\frac{1}{2}}} e^{-\frac{(Y_0-\bar{x}_0)'(\bar{V}_0)^{-1}(Y_0-\bar{x}_0)}{2}} \\ &\times \frac{1}{(2\pi)^{\frac{d}{2}}(\det(B\Sigma_W B'))^{\frac{1}{2}}} e^{-\frac{(Y_1-Y_0)'(B\Sigma_W B')^{-1}(Y_1-Y_0)}{2}} \\ &\times \cdots \times \frac{1}{(2\pi)^{\frac{d}{2}}(\det(B\Sigma_W B'))^{\frac{1}{2}}} \\ &\times e^{-\frac{(Y_{T-1}-Y_{T-2})'(B\Sigma_W B')^{-1}(Y_{T-1}-Y_{T-2})}{2}}. \end{aligned}$$

Hence $\int \log\left(g_{Y^{T-1}}(y^{T-1})\right) f_{Y^{T-1}}(y^{T-1})dy^{T-1} = -\frac{Td}{2}\log(2\pi) - \frac{T-1}{2}\log\det(B\Sigma_W B') - \frac{1}{2}trac\left((\bar{V}_0)^{-1}E[(Y_0-\bar{x}_0)(Y_0-\bar{x}_0)']\right) - \frac{1}{2}\sum_{t=0}^{T-2} trac\left((B\Sigma_W^{-1}B')^{-1}E[BW_t(BW_t)']\right) - \frac{1}{2}\log\det\bar{V}_0 = -\frac{Td}{2}\log(2\pi e) - \frac{T-1}{2}\log\det(B\Sigma_W B') - \frac{1}{2}\log\det\bar{V}_0$. Let $K(\bar{W}^{T-2}) \triangleq \frac{1}{2T}E[\sum_{t=0}^{T-2}\bar{W}_t'\Sigma_W^{-1}\bar{W}_t] - R_c - E[\frac{1}{2T}\sum_{t=0}^{T-1}Y_t'MY_t]$ and $M(\bar{W}^{T-2}) \triangleq \frac{1}{2T}E[\sum_{t=0}^{T-2}\bar{W}_t'\Sigma_W^{-1}\bar{W}_t] - \frac{d}{2}\log(2\pi e) - \frac{1}{2T}\log\det\bar{V}_0 - \frac{T-1}{2T}\log\det(B\Sigma_W B')$. Then, from (25), it follows that $\sup_{f_{Y^{T-1}}\in\mathcal{D}_{SU}(g_{Y^{T-1}})}\frac{1}{T}H_S(f_{Y^{T-1}}) = \sup_{\{\{\bar{W}_t\}_{t=0}^{T-2};K(\bar{W}^{T-2})\le 0\}} -M(\bar{W}^{T-2}) = -\inf_{\{\{\bar{W}_t\}_{t=0}^{T-2};K(\bar{W}^{T-2})\le 0\}} M(\bar{W}^{T-2})$. Since the conditions for applying the Lagrange duality theorem (Luenberger, 1969) are satisfied, by applying this theorem we have the following:

$$\frac{1}{T}\sup_{f_{Y^{T-1}}\in\mathcal{D}_{SU}(g_{Y^{T-1}})} H_S(f_{Y^{T-1}}) = \min_{s\ge 0}\left\{sR_c + \frac{d}{2}\log(2\pi e) + \frac{1}{2T}\log\det\bar{V}_0 + \frac{T-1}{2T}\log\det(B\Sigma_W B')\right.$$

$$- \inf_{\{\bar{W}_t\}_{t=0}^{T-2}} \frac{1}{2T} E\left[\sum_{t=0}^{T-2}\left[Y_t'(-sM)Y_t + \bar{W}_t'\left((1+s)\Sigma_W^{-1}\right)\bar{W}_t\right] + Y_{T-1}'(-sM)Y_{T-1}\right]\Bigg\}, \quad (26)$$

where $s > 0$ is the Lagrange multiplier. Next, by applying the stochastic dynamic programming (Caines, 1988, pp. 670–677), the solution to the above optimization problem is obtained. Toward this goal, define the following value function: $J(t,Y) \triangleq \inf_{\{\bar{W}_k\}_{k=t}^{T-2}} E[\sum_{k=t}^{T-2}[\bar{W}_k'(1+s)\Sigma_W^{-1}\bar{W}_k + Y_k'(-sM)Y_k] + Y_{T-1}'(-sM) Y_{T-1}|Y_t]$. This value function satisfies the following stochastic dynamic programming (Caines, 1988):

$$J(t,Y) = \inf_{\bar{W}_t} E[\bar{W}_t'(1+s)\Sigma_W^{-1}\bar{W}_t + Y_t'(-sM)Y_t + J(t+1,Y)|Y_t]. \quad (27)$$

Next, we pick up the following candidate $J(t,Y) = -Y_t'\Xi_t Y_t - \Theta_t$ (Ξ_t is symmetric) with the final condition $J(T-1,Y) \triangleq -Y_{T-1}'\Xi_{T-1}Y_{T-1} - \Theta_{T-1} = Y_{T-1}'(-sM)Y_{T-1}$ as the solution of the dynamic programming (27). Hence, from (27), we have

$$-Y_t'\Xi_t Y_t - \Theta_t = \inf_{\bar{W}_t} E\left[\bar{W}_t'(1+s)\Sigma_W^{-1}\bar{W}_t + Y_t'(-sM)Y_t - Y_{t+1}'\Xi_{t+1}Y_{t+1} - \Theta_{t+1}|Y_t\right],$$

$$(Y_{t+1} = AY_t + BW_t + B\bar{W}_t). \quad (28)$$

The solution to the above optimization problem is given by (provided the inequality $(1+s)\Sigma_W^{-1} > B'\Xi_{t+1}B$ holds)

$$\bar{W}_t^* = -\left(-(1+s)\Sigma_W^{-1} + B'\Xi_{t+1}B\right)^{-1} B'\Xi_{t+1}AY_t. \quad (29)$$

Therefore, by substituting (29) in (28), we have the following recursions for Ξ_t and Θ_t: $\Xi_t = A'\Xi_{t+1}A - A'\Xi_{t+1}B(-(1+s)\Sigma_W^{-1} + B'\Xi_{t+1}B)^{-1}B'\Xi_{t+1}A + sM$, $\Xi_{T-1} = sM$, and $\Theta_t = \Theta_{t+1} + trac(B'\Xi_{t+1}B\Sigma_W)$, $\Theta_{T-1} = 0$. From our assumptions, it follows that Ξ_t converges to Ξ_∞, the unique stabilizing solution of Eq. (10). Subsequently, from the following equality: $\inf_{\{\bar{W}_t\}_{t=0}^{T-2}} E[\sum_{t=0}^{T-2}[Y_t'(-sM)Y_t + \bar{W}_t'((1+s)\Sigma_W^{-1})\bar{W}_t] + Y_{T-1}'(-sM)Y_{T-1}] = E[J(0,Y)] = E[Y_0'\Xi_0 Y_0 - \Theta_0]$, and by letting $T \to \infty$ in (26), the robust entropy rate is obtained, and is given by (11). □

Proof of Corollary 4.2. (i) For each $t \geq 0$, we have $E[K_t^2] \leq \Psi_t$ for all subsystems. Therefore, $E[K_t - \tilde{K}_t]^2 = (1-\gamma_t\alpha_t)^2 E[K_t^2] + \gamma_t^2 E[\tilde{W}_t^2] = \frac{D^2}{\Psi_t^2}E[K_t^2] + D(1-\frac{D}{\Psi_t}) \leq \frac{D^2}{\Psi_t^2}\Psi_t + D - \frac{D^2}{\Psi_t} = D$. Moreover, $E[Y_t - \tilde{Y}_t]^2 = E[Y_t - \hat{X}_t - \tilde{Y}_t + \hat{X}_t]^2 = E[(Y_t - \hat{X}_t) - (\tilde{Y}_t - \hat{X}_t)]^2 = E[K_t - \tilde{K}_t]^2 \leq D$.

(ii) The capacity of an AWGN channel, as described in Section 2, is given by (Cover & Thomas, 1991)

$$\mathcal{C} = \lim_{T\to\infty} \frac{1}{2T}\sum_{t=0}^{T-1}\log\left(1 + \frac{P_t}{W_c}\right), \quad (30)$$

where W_c is the variance of the channel noise $\tilde{W}_t \sim N(0, W_c)$. By substituting $P_t = \frac{\eta_t W_c}{D}\Psi_t$ in (30), we have $\mathcal{C} = \lim_{T\to\infty}\frac{1}{2T}\sum_{t=0}^{T-1}\log(1 + \frac{\eta_t\Psi_t}{D}) = \frac{1}{2}\log(1 + \frac{(1-\frac{D}{\Psi_\infty})\Psi_\infty}{D}) = \frac{1}{2}\log\frac{\Psi_\infty}{D}$ ($\Psi_\infty = \lim_{t\to\infty}\Psi_t$). On the other hand, for τ sufficiently large, the sequence $\{K_t\}_{t\in\mathbf{N}_+}$ corresponding to the subsystem with maximum entropy is an orthogonal Gaussian sequence with distribution $K_t \sim N(0,\Psi_t)$. Therefore, $\mathcal{H}_r(\mathcal{K}) = \lim_{T\to\infty}\frac{1}{2T}\log\left[(2\pi e)^T\prod_{t=0}^{T-1}\Psi_t\right] = \frac{1}{2}\log(2\pi e\Psi_\infty)$. Hence, the lower bound (7) for $d = 1, r = 2$ and the sequence $\{K_t\}_{t\in\mathbf{N}_+}$ (as described above) is equal to $\mathcal{H}_r(\mathcal{K}) - \frac{1}{2}\log(2\pi eD) = \frac{1}{2}\log(2\pi e\Psi_\infty) - \frac{1}{2}\log(2\pi eD) = \frac{1}{2}\log\frac{\Psi_\infty}{D} = \mathcal{C}$. Therefore, following the necessary condition (7), $\mathcal{C} = R_{SRD,r}^{K,\tilde{K}}(D) = \mathcal{H}_r(\mathcal{K}) - \frac{1}{2}\log(2\pi eD) = \frac{1}{2}\log\frac{\Psi_\infty}{D}$ is the minimum required capacity for uniform mean-square reconstructability of the message K_t by $\tilde{K}_t$.

Next, we show that the sequential rate distortion functions associated with the sequences $\{K_t\}_{t\in\mathbf{N}_+}$ and $\{Y_t\}_{t\in\mathbf{N}_+}$ are the same. This equality implies that the necessary condition (7) is also tight for uniform mean-square reconstructability of the observation sequence.

To achieve this goal, consider the following sequence of equalities: $I(Y^{T-1} \to \tilde{Y}^{T-1}) = \sum_{t=0}^{T-1} H_S(\tilde{Y}_t|\tilde{Y}^{t-1}) - \sum_{t=0}^{T-1} H_S(\tilde{Y}_t|\tilde{Y}^{t-1}, Y^t) = \sum_{t=0}^{T-1} H_S(\tilde{K}_t + C\hat{X}_t|\tilde{K}_{-1} + C\hat{X}_{-1}, \ldots, \tilde{K}_{t-1} + C\hat{X}_{t-1}) - \sum_{t=0}^{T-1} H_S(\tilde{K}_t + C\hat{X}_t|\tilde{K}_{-1} + C\hat{X}_{-1}, K_0 + C\hat{X}_0, \ldots, \tilde{K}_{t-1} + C\hat{X}_{t-1}, K_t + C\hat{X}_t) = \sum_{t=0}^{T-1}\left(H_S(\tilde{K}_t|\tilde{K}^{t-1}) - H_S(\tilde{K}_t|\tilde{K}^{t-1}, K^t)\right) = I(K^{T-1} \to \tilde{K}^{T-1})$, where the first equality follows from the definition (see Cover & Thomas, 1991, p. 231), the second follows by substituting the expression of Y_t and $\tilde{Y}_t$, the third by variant of (Cover & Thomas, 1991, Theorem 9.6.3) and the fourth by definition. Note that $\tilde{K}_{-1}$ can be taken to be zero and $\hat{X}_{-1} = \hat{X}_0$. Hence, using the equalities $I(Y^{T-1} \to \tilde{Y}^{T-1}) = I(K^{T-1} \to \tilde{K}^{T-1})$ and $E\|Y_t - \tilde{Y}_t\|^2 = E\|K_t - \tilde{K}_t\|^2$, we can conclude that the robust sequential rate distortion functions (see (5)) $R_{SRD,r}^{Y,\tilde{Y}}(D)$ and $R_{SRD,r}^{K,\tilde{K}}(D)$ subject to the single letter mean-square distortion criterion are the same. Therefore, from Corollary 4.2(i) and Theorem 3.1 it follows that $\mathcal{C} = R_{SRD,r}^{Y,\tilde{Y}}(D) = R_{SRD,r}^{K,\tilde{K}}(D) = \mathcal{H}_r(\mathcal{K}) - \frac{1}{2}\log(2\pi eD) = \frac{1}{2}\log\frac{\Psi_\infty}{D}$ is also the minimum required capacity for uniform mean-square reconstructability of the observation sequence. □

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