Stability and Tracking of Linear Gaussian Systems over AWGN Channel with Intermittent Deterministic Feedback Channel

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***Abstract*—This paper shows that innovation encoder, the Kalman filter decoder and certainty equivalent controller are still good choices for stability and tracking of linear Gaussian systems over Additive White Gaussian Noise (AWGN) channel in the presence of intermittent deterministic noiseless feedback channel. When feedback channel is described by a deterministic switch, for both stable and unstable systems, an innovation encoder, the Kalman filter decoder and a certainty equivalent controller are presented for stability and tracking over AWGN channel provided channel input power constraint is not violated and the system is controllable. Then, it is illustrated that when feedback channel is frequently available, performance of the system is similar to the optimal performance which corresponds to the case of full time availability of noiseless feedback channel.**

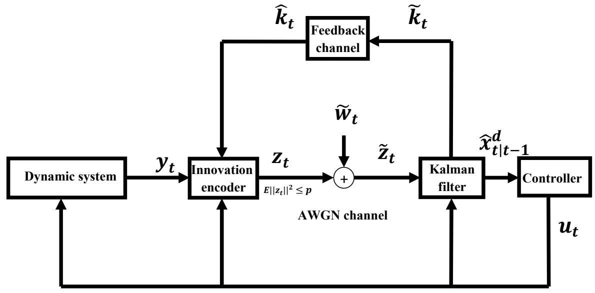

Fig. 1. Linear Gaussian system over AWGN channel.

I. INTRODUCTION

One of the issues that has begun to emerge in a number of applications is how to track state trajectory of a dynamic system at a remote controller and control it over a communication channel subject to imperfections (e.g., transmission noise). Some examples of systems that are required to be tracked and controlled over communication channels subject to imperfections are smart drilling system and smart oil well. Some results addressing basic problems in tracking and/or stability of dynamic systems over communication channels subject to imperfections can be found in [1]-[12].

Dynamic systems can be viewed as continuous alphabet sources with memory. Consequently, many works in the literature (e.g., [1],[10], [11],[12]) including this paper are dedicated to the question of stability and tracking over Additive White Gaussian Noise (AWGN) channel, which itself is naturally a continuous alphabet channel. In [10] the authors considered the problem of stability and tracking of partially observed discrete time linear Gaussian systems over AWGN channel and for the quadratic cost functional, they showed that an innovation encoder that exploits noiseless feedback channel full time, the Kalman filter decoder and a certainty equivalent controller minimize the quadratic cost functional (with the optimal cost of J_{opt}) and results in real time reliable data reconstruction up to a given distortion level (i.e., tracking) and stability. [11] includes the fully observed version of the results of [10] and [12] presents the continuous time fully observed version of [10].

As the availability of noiseless feedback channel may not be possible full time, it is more desirable to use noiseless feedback channel that is available intermittently. Therefore, this paper aims to investigate if innovation encoder, the Kalman filter decoder and certainty equivalent controller are still good choices for stability and tracking of linear Gaussian systems over AWGN channel in the presence of noiseless feedback channel that is available intermittently.

To address this question, when feedback channel is described by a deterministic switch, for both stable and unstable systems, an innovation encoder, the Kalman filter decoder and a certainty equivalent controller are presented for stability and tracking over AWGN channel provided channel input power constraint is not violated and the system is controllable. Then, it is illustrated that when feedback channel is frequently available, performance of the system is similar to the optimal performance which corresponds to the case of full time availability of noiseless feedback channel [10].

The paper is organized as follows. In Section II, the problem formulation is presented. In Section III, an innovation encoder, the Kalman filter decoder and a certainty equivalent controller are presented for stability and tracking when feedback channel is described by a deterministic switch. Section IV is devoted to simulation studies, and in Section V the paper is concluded by summarizing the contributions of the paper.

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II. PROBLEM FORMULATION

Throughout, the following conventions are used. $\mathbb{R}$ denotes the space of real numbers, I the identity matrix, $||\cdot||$ the Euclidean norm and $\doteq$ means 'by definition is equivalent to'. $y^t \doteq (y_0, y_1, ..., y_t)$, $N(m,n)$ denotes the Gaussian distribution with mean m and variance n, $var(k)$ denotes the variance of the random variable k and $E[k]$ is the expected value. $trac(A)$ denotes the trace of square matrix A, A^{-1} the inverse of square matrix A, V' the transpose of matrix/vector V and $f_{\tilde{k}|k}$ denotes the conditional density function.

This paper investigates if innovation encoder, the Kalman filter decoder and certainty equivalent control are still good choices for stability and tracking of linear Gaussian systems over AWGN channel in the presence of intermittent deterministic feedback channel as described in the block diagram of Fig. 1. This block diagram with full time availability of feedback channel has been considered in several research papers, such as [10], [12], [11]. This block diagram can describe the tele-operation problem of micro vehicles, such as micro Unmanned Aerial Vehicles (UAVs) and micro Autonomous Underwater Vehicles (AUVs). These micro vehicles are supplied with limited capacity batteries. Consequently, the communication from these vehicles to base station where controller is located is subject to imperfections, which are modeled in the block diagram of Fig. 1 by AWGN channel. However, as the base station can be equipped with high capacity power supplies, the communication from controller to these vehicles can be assumed without imperfections.

The building blocks of the block diagram of Fig. 1 are described now:

Dynamic System: The dynamic system is the following partially observed linear Gaussian system

$$\begin{cases} x_{t+1} = Ax_t + Hu_t + Bw_t, \\ y_t = Cx_t + Dv_t, \end{cases} \quad (1)$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^o$, $w_t \in \mathbb{R}^q$, $y_t \in \mathbb{R}$, $v_t \in \mathbb{R}^l$, $A \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{n\times o}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}^{1\times l}$. w_t i.i.d. $\sim N(0,I)$, v_t i.i.d. $\sim N(0,I)$, $x_0 \sim N(\bar{x}_0, Q_0)$, and w_t, v_t, x_0 are mutually independent.

Encoder: Encoder is innovation encoder that transmits an error between message y_t and an estimate of the reconstruction of y_t at the end of communication. That is, if $\mathcal{E}(\cdot)$ denotes the encoder operation, then $z_t = \mathcal{E}(y^t, u^{t-1}, \hat{k}^{t-1}) = \alpha_t k_t$, where $k_t = y_t - y_t^e$ and $\alpha_t \in \mathbb{R}$. Note that y_t^e is an estimate of y_t and $\hat{k}_t$ is the feedback channel signal that is available at the encoder. u_t is the control signal.

Communication Channel: The communication channel is the Additive White Gaussian Noise (AWGN) channel with input z_t and output $\tilde{z}_t$ such that $\tilde{z}_t = z_t + \tilde{w}_t$, where $\tilde{w}_t$ i.i.d. $\sim N(0, w_c)$ is the channel noise. $\tilde{w}_t$ is independent of w_t, v_t, x_0. This channel is subject to channel input power constraint of $E||z_t||^2 \leq p$.

Decoder: Decoder is the Kalman filter (i.e., the minimum mean square error estimator) with input $\tilde{k}^{t-1}$ ($\tilde{k}_t = \gamma_t \tilde{z}_t$, $\gamma_t \in \mathbb{R}$) and u^{t-1}, and output $\hat{x}^d_{t|t-1}$, where $\hat{x}^d_{t|t-1}$ is the mean square state estimation at time instant t given $(\tilde{k}^{t-1}, u^{t-1})$. Note that $\tilde{k}_t$ is the feedback channel signal.

Controller: Controller is a certainty equivalent controller of the following form: $u_t = K\hat{x}^d_{t|t-1}$, where $K \in \mathbb{R}^{o\times n}$ is the controller gain. Note that in this paper it is assumed that the encoder cannot estimate $\hat{x}^d_{t|t-1}$ by observing the control signal u_t over time (for example, $K'K$ is not invertible).

Intermittent Deterministic Feedback Channel: Feedback channel is described by a switch, in which its schedule for being on or off is known a priori for both the encoder and decoder. The full time availability of noiseless feedback channel requires that the transmission of feedback channel signal $\tilde{k}_t$ is done with high power full time. In the block diagram of Fig. 1, the control signal u_t is also transmitted with high power. Hence, the availability of noiseless feedback channel full time results in significant power consumption at the receiver. Therefore, it is more desirable to transmit the feedback channel signal with high power only at specific time instants that are known for both the encoder and decoder to avoid wasting the receiver power supply. This motivates the use of intermittent deterministic feedback channel.

The objective of this paper is to investigate if innovation encoder, the Kalman filter decoder and certainty equivalent controller are still good choices for stability and tracking of the block diagram of Fig. 1, as described as follows.

Definition 2.1: (Tracking): Consider the block diagram of Fig. 1 described by the dynamic system (1). The system (1) is tractable if and only if there exist an innovation encoder and the Kalman filter decoder that result in real-time reliable data reconstruction up to the given distortion level $\bar{D} \geq 0$, as follows: $E||y_t - \tilde{y}_t||^2 \leq \bar{D}$, where $\tilde{y}_t$ is the reconstruction of y_t at the end of communication.

Definition 2.2: (Stability): Consider the block diagram of Fig. 1 described by the dynamic system (1). The system (1) is stabilizable if and only if there exist an innovation encoder, the Kalman filter decoder and a certainty equivalent controller that result in stability of the dynamic system in the following form

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} E||x_t||^2_Q < \infty$$

by minimizing the following cost functional

$$J = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} E(||x_t||^2_Q + ||u_t||^2_R),$$
$$Q = Q' \geq 0, \; R = R' > 0. \quad (2)$$

III. INTERMITTENT DETERMINISTIC FEEDBACK CHANNEL

Consider the block diagram of Fig. 1 described by the intermittent deterministic feedback channel, as described earlier. In this section, it is assumed that in the first M time instants, there is noiseless feedback channel; but in the next $N-M$ time instants ($N \geq M$), feedback channel is not available. Similarly, in time instants $N+1$, $N+2$, ..., $N+M$, noiseless feedback channel is available; but in time instants $N+M+1$, ..., $2N$, feedback channel is not available; and so on and so forth. Under this assumption, in this section, we present an innovation encoder, the Kalman filter decoder and a certainty equivalent controller that result in stability and tracking, as defined earlier.

Innovation Encoder: For those time instants that feedback channel is available, the encoder has the following description.

$$\begin{aligned} k_t &= y_t - E[y_t|\tilde{k}_{nN}, \tilde{k}_{nN+1}, ..., \tilde{k}_{t-1}, u^{t-1}], \\ &\quad t \in \{nN, nN+1, ..., nN+M\}, \\ &\quad n = \{0,1,2,3,...\} \\ &= C(x_t - \hat{x}^e_{t|t-1}) + Dv_t, \\ \hat{x}^e_{t|t-1} &= E[x_t|\tilde{k}_{nN}, \tilde{k}_{nN+1}, ..., \tilde{k}_{t-1}, u^{t-1}] \\ z_t &= \alpha_t k_t. \end{aligned}$$

Therefore, the dynamic system that is seen by the encoder with the measurement $\bar{y}_t = \tilde{k}_t + \alpha_t\gamma_t C\hat{x}^e_{t|t-1}$ (α_t and γ_t will be defined shortly) is the following dynamic system.

$$\begin{cases} x_{t+1} = Ax_t + Hu_t + Bw_t, \\ \bar{y}_t = \alpha_t\gamma_t Cx_t + \alpha_t\gamma_t Dv_t + \gamma_t\tilde{w}_t. \end{cases} \quad (3)$$

Hence, from the standard results of Kalman filtering [13] it follows for $t \in \{nN, nN+1, ..., nN+M-1\}$ that

$$\hat{x}^e_{t+1|t} = A\hat{x}^e_{t|t-1} + Hu_t + A\Delta^e_t\tilde{k}_t, \quad \hat{x}^e_{0|-1} = \bar{x}_0, \quad (4)$$

$$\begin{aligned} \Delta^e_t &= P^e_{t|t-1}\alpha_t\gamma_t C'(\alpha_t^2\gamma_t^2 CP^e_{t|t-1}C' \\ &\quad +\alpha_t^2\gamma_t^2 DD' + \gamma_t^2 w_c)^{-1} \\ P^e_{t+1|t} &= AP^e_{t|t-1}A' - A\Delta^e_t\alpha_t\gamma_t CP^e_{t|t-1}A' \\ &\quad +BB', \quad P^e_{0|-1} = Q_0. \end{aligned} \quad (5)$$

But, for those time instants that feedback channel is not available, the encoder uses an open loop estimation. That is,

$$\begin{aligned} k_t &= y_t - E[y_t|y_{nN+M}, y_{nM+M+1}, ..., y_{t-1}, u^{t-1}], \\ &\quad t \in \{nN+M, ..., nN+N\} \\ &= C(x_t - \hat{x}^e_{t|t-1}) + Dv_t, \\ &\quad \hat{x}^e_{t|t-1} = E[x_t|y_{nN+M}, ..., y_{t-1}, u^{t-1}]. \\ z_t &= \alpha_t k_t. \end{aligned}$$

Hence, for $t \in \{nN+M, ..., nN+N-1\}$ we have the following recursive equations.

$$\hat{x}^e_{t+1|t} = A\hat{x}^e_{t|t-1} + Hu_t \quad (6)$$

$$\begin{aligned} P^e_{t+1|t} &\doteq E[\tilde{x}^e_{t+1|t}\tilde{x}^{e'}_{t+1|t}] = AP^e_{t|t-1}A' + BB' \\ \tilde{x}^e_{t+1|t} &= A\tilde{x}^e_{t|t-1} + Bw_t \quad (\tilde{x}^e_{t|t-1} = x_t - \hat{x}^e_{t|t-1}) \\ &. \end{aligned} \quad (7)$$

Remark 3.1: Note that for $n \geq 1$, the initial condition for the recursive equation (4), i.e., $\hat{x}_{nN|nN-1}$ is the estimate of the state that is obtained from the recursive equation (6) for $t = nN-1$. Also, the initial condition for the recursive equation (6), i.e., $\hat{x}_{nN+M|nN+M-1}$ is the estimate of the state that is obtained from the recursive equation (4) for $t = nN+M-1$. Consequently, for $n \geq 1$, the initial condition for the recursive equation (5), i.e., $P^e_{nN|nN-1}$ is the mean square estimation error that is obtained from the recursive equation (7) for $t = nN-1$. Also, the initial condition for the recursive equation (7) is the mean square estimation error that is obtained from the recursive equation (5) for $t = nN+M-1$.

The Kalman Filter Decoder: As for all time instants we have $\tilde{k}_t = \alpha_t\gamma_t k_t + \gamma_t\tilde{w}_t = \alpha_t\gamma_t C(x_t - \hat{x}^e_{t|t-1}) + \alpha_t\gamma_t Dv_t + \gamma_t\tilde{w}_t$, the dynamic system that is seen by the decoder is the following

$$\begin{cases} x_{t+1} = Ax_t + Hu_t + Bw_t, \\ \tilde{k}_t = \alpha_t\gamma_t C(x_t - \hat{x}^e_{t|t-1}) + \alpha_t\gamma_t Dv_t + \gamma_t\tilde{w}_t. \end{cases} \quad (8)$$

Note that as the feedback channel is deterministic, $\hat{x}^e_{t|t-1}$ is known to the decoder for all time instants. Consequently, the dynamic system with the measurement $\bar{k}_t \doteq \tilde{k}_t + \alpha_t\gamma_t C\hat{x}^e_{t|t-1}$ that is seen by the decoder is the following

$$\begin{cases} x_{t+1} = Ax_t + Hu_t + Bw_t, \\ \bar{k}_t = \alpha_t\gamma_t Cx_t + \alpha_t\gamma_t Dv_t + \gamma_t\tilde{w}_t. \end{cases} \quad (9)$$

Therefore, from the standard results of Kalman filtering [13] it follows that

$$\begin{aligned} \hat{x}^d_{t+1|t} &= A\hat{x}^d_{t|t-1} + Hu_t + A\Delta^d_t(\tilde{k}_t + \alpha_t\gamma_t C\hat{x}^e_{t|t-1} \\ &\quad -\alpha_t\gamma_t C\hat{x}^d_{t|t-1}) \\ &= A\hat{x}^d_{t|t-1} + Hu_t + A\Delta^d_t(\alpha_t\gamma_t C\tilde{x}^d_{t|t-1} \\ &\quad +\alpha_t\gamma_t Dv_t + \gamma_t\tilde{w}_t), \quad \hat{x}^d_{0|-1} = \bar{x}_0 \\ \Delta^d_t &= P^d_{t|t-1}\alpha_t\gamma_t C'(\alpha_t^2\gamma_t^2 CP^d_{t|t-1}C' \\ &\quad +\alpha_t^2\gamma_t^2 DD' + \gamma_t^2 w_c)^{-1} \\ P^d_{t+1|t} &= AP^d_{t|t-1}A' - A\Delta^d_t\alpha_t\gamma_t CP^d_{t|t-1}A' + BB', \\ &\quad P^d_{0|-1} = Q_0 \end{aligned} \quad (10)$$

$$\begin{aligned} &\tilde{x}^d_{t+1|t} = (A - A\Delta^d_t\alpha_t\gamma_t C)\tilde{x}^d_{t|t-1} - A\Delta^d_t\alpha_t\gamma_t Dv_t \\ &-A\Delta^d_t\gamma_t\tilde{w}_t + Bw_t \ (\tilde{x}^d_{t|t-1} = x_t - \hat{x}^d_{t|t-1}). \end{aligned} \quad (11)$$

Determination of α_t and γ_t: α_t and γ_t are determined from the Shannon source-channel matching principle which requires that the message to reconstructed message behaves like the rate distortion minimizing stochastic kernel. In the block diagram of Fig. 1, let k_t be the message and $\tilde{k}_t$ the reconstructed message. Then, as $\tilde{k}_t = \alpha_t\gamma_t k_t + \gamma_t\tilde{w}_t$, we have $f_{\tilde{k}_t|k_t} \sim N(\alpha_t\gamma_t k_t, \gamma_t^2 w_c)$. On the other hand, for Gaussian random variable k_t

with mean square error distortion measure, the rate distortion minimizing stochastic kernel is $f^*_{\tilde{k}_t|k_t} \sim N(\eta_t k_t, \eta_t \bar{D})$, where $\bar{D}(\leq \psi_t)$ is the given distortion level, $\psi_t = var(k_t)$ and $\eta_t = 1 - \frac{\bar{D}}{\psi_t}$ [14]. Now, to fulfill the requirement of Shannon source-channel matching principle, α_t and γ_t must be defined such that $f_{\tilde{k}_t|k_t} = f^*_{\tilde{k}_t|k_t}$ resulting in

$$\alpha_t = \sqrt{\frac{\eta_t w_c}{\bar{D}}}, \quad \gamma_t = \sqrt{\frac{\eta_t \bar{D}}{w_c}}, \quad \bar{D} \leq \psi_t. \tag{12}$$

Now, we have the following proposition for tracking using the proposed coding technique.

Proposition 3.2: Consider the block diagram of Fig. 1 described by the intermittent deterministic feedback channel and the proposed coding technique with α_t and γ_t as given in (12). Suppose that the channel input power constraint is not violated. Then, we have real-time tracking as follows: $E||y_t - \tilde{y}_t||^2 = \bar{D}$, where $\tilde{y}_t = \tilde{k}_t + C\hat{x}^e_{t|t-1}$.

Proof: Using the proposed coding technique we have the following equalities:

$$\begin{aligned} E||k_t - \tilde{k}_t||^2 &= E||(1-\alpha_t\gamma_t)k_t - \gamma_t \tilde{w}_t||^2 \\ &= E||\frac{\bar{D}}{\psi_t}k_t - \gamma_t \tilde{w}_t||^2 \\ &= E[\frac{\bar{D}^2}{\psi_t^2}k_t^2 + \gamma_t^2 \tilde{w}_t^2 - 2\frac{\bar{D}}{\psi_t}\gamma_t k_t \tilde{w}_t] \\ &= \frac{\bar{D}^2}{\psi_t^2}E[k_t^2] + \gamma_t^2 E[\tilde{w}_t^2] + 0 = \bar{D}. \end{aligned}$$

Now, as $E||y_t - \tilde{y}_t||^2 = E||k_t - \tilde{k}_t||^2$, we have $E||y_t - \tilde{y}_t||^2 = \bar{D}$. This completes the proof.

Under the assumption that the channel input power constraint is not violated, from the descriptions for encoding error and decoding error (11) and the standard results of Kalman filtering, it follows that the proposed coding technique is stable even for unstable matrix A. Note that by stability of coding technique we mean that the mean square encoding estimation error $E||\tilde{x}^e_{t|t-1}||^2$ and mean square decoding estimation error $E||\tilde{x}^d_{t|t-1}||^2$ are asymptotically bounded.

Now, using the proposed coding technique and under the assumptions that the channel input power constraint is not violated, the static controller gain K of the certainty equivalent controller $u_t = K\hat{x}^d_{t|t-1}$ is determined by minimizing the cost functional (2). To achieve this goal we notice that the dynamic system that is seen in the controller side is the system (9). Now, to determine the static controller gain K, we apply the separation principle [13]. That is, we rewrite the cost functional (2) as follows

$$\begin{aligned} J &= \lim_{T\to\infty}\frac{1}{T}E[\sum_{t=0}^{T-1}(x_t - \hat{x}^d_{t|t-1} + \hat{x}^d_{t|t-1})' \\ &\quad .Q(x_t - \hat{x}^d_{t|t-1} + \hat{x}^d_{t|t-1}) + u_t'Ru_t] \\ &= \lim_{T\to\infty}\frac{1}{T}E[\sum_{t=0}^{T-1}(\tilde{x}^d_{t|t-1} + \hat{x}^d_{t|t-1})' \\ &\quad .Q(\tilde{x}^d_{t|t-1} + \hat{x}^d_{t|t-1}) + u_t'Ru_t]. \end{aligned} \tag{13}$$

Now, as the estimation error is orthogonal to the observation space, that is, $E[\tilde{x}^{d'}_{t|t-1}\hat{x}^d_{t|t-1}] = 0$ and $E[\hat{x}^{d'}_{t|t-1}\tilde{x}^d_{t|t-1}] = 0$, we have the following equalities

$$\begin{aligned} J &= \lim_{T\to\infty}\frac{1}{T}E[\sum_{t=0}^{T-1}\tilde{x}^{d'}_{t|t-1}Q\tilde{x}^d_{t|t-1} + \hat{x}^{d'}_{t|t-1}Q\hat{x}^d_{t|t-1} \\ &\quad +u_t'Ru_t] \\ &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}trac(QP^d_{t|t-1}) + \bar{J}, \\ &\quad \bar{J} \doteq \lim_{T\to\infty}\frac{1}{T}E[\sum_{t=0}^{T-1}\hat{x}^{d'}_{t|t-1}Q\hat{x}^d_{t|t-1} + u_t'Ru_t]. \end{aligned} \tag{14}$$

From the equality (14) it is evident that the control input $u_t = K\hat{x}^d_{t|t-1}$ that minimizes the cost functional (2) is the one that minimizes the cost functional $\bar{J}$ as $P^d_{t|t-1}$ is independent of the control input. From the standard results of stochastic linear quadratic regulator [13] it follows that under the assumption that the pair (A, H) is controllable, the control input that minimizes the cost $\bar{J}$ subject to (10) is given by

$$\begin{aligned} u_t &= K\hat{x}^d_{t|t-1} \\ K &= -(R + H'P_\infty H)^{-1}H'P_\infty A, \end{aligned} \tag{15}$$

where $P_\infty = P_\infty' > 0$ is the solution of the following Algebraic Riccati equation.

$$\begin{aligned} P_\infty &= A'P_\infty A - A'P_\infty H(R + H'P_\infty H)^{-1}H'P_\infty A \\ &\quad +Q. \end{aligned}$$

Remark 3.3: From the standard results of stochastic linear quadratic regulator [13] it follows that $A + HK$ is a stable matrix.

Now, by substituting $u_t = K\hat{x}^d_{t|t-1}$ in the cost functional (2) the minimum cost functional is calculated as follows.

$$\begin{aligned} J^* &= \lim_{T\to\infty}\frac{1}{T}\{E[\sum_{t=0}^{T-1}trac(QP^d_{t|t-1})] \\ &\quad +E[trac(P_0(Q_0 + \bar{x}_0\bar{x}_0')) \\ &\quad +\sum_{t=0}^{T-1}trac(P_{t+1}\Gamma_t)]\}, \end{aligned} \tag{16}$$

where P_t is the solution of the following backward in time Riccati equation.

$$\begin{aligned} P_{t+1} &= A'P_tA - A'P_tH(R + H'P_tH)^{-1}H'P_tA \\ &\quad +Q, \ P_T = 0, \end{aligned}$$

and

$$\Gamma_t \doteq E[A\Delta^d_t\tilde{k}_t\tilde{k}_t'\Delta^{d'}_t A'].$$

Now, in the following proposition we show that using the proposed coding technique and controller the system (1) in the block diagram of Fig. 1 is stable.

Proposition 3.4: Consider the block diagram of Fig. 1 described by the dynamic system (1). Suppose that the pair (A, H) is controllable and the channel input power constraint is not violated. Then, using the proposed coding technique and controller the system (1) in the block diagram of Fig. 1 is stable.

Proof: Under the action of the proposed controller, the dynamic system has the following representation

$$\begin{aligned} x_{t+1} &= Ax_t + Hu_t + Bw_t \\ &= Ax_t + HK\hat{x}^d_{t|t-1} + Bw_t \\ &= (A + HK)x_t - HK\tilde{x}^d_{t|t-1} + Bw_t. (17) \end{aligned}$$

Now, as the matrix $A + HK$ is stable and $\lim_{t\to\infty} E||\tilde{x}^d_{t|t-1}||^2 < \infty$, from the dynamic system (17) it follows that $\lim_{t\to\infty} E||x_t||^2_Q < \infty$. This inequality results in the stability of the following form $\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} E||x_t||^2_Q < \infty$.

IV. SIMULATION RESULTS

In this section, the sub-optimality of the proposed coding technique and controller are illustrated by computer simulations.

Consider the block diagram of Fig. 1 described by AWGN channel, the proposed coding technique and controller and a scalar dynamic system with the following specification: $w_c = 1$, $p = 30$, $x_0 \sim N(10, 1)$, $H = B = C = D = 1$. For unstable dynamic system, we set $A = 1.2$ but for stable dynamic, we set $A = 0.8$. For simulation study, we also set $\bar{D} = 0.5$, $Q = R = 1$ and $N = 10$.

Fig. 2 illustrates the trade-off between J^* and M when the block diagram of Fig. 1 is described by the stable dynamic system with $N = 10$. As clear from this figure for all $M \in \{1, 2, ..., 10\}$, the difference between J^* and $J_{opt} = J^*\Big|_{M=N}$ is negligible. This indicates that for the condition simulated, almost noting is lost in terms of performance if feedback channel is used intermittently. Fig. 3 also illustrates the trade-off between J^* and M when the block diagram of Fig. 1 is described by the unstable dynamic system. Note that for the unstable dynamic system $J_{opt} = 6.68$. For this case, it is observed for $M \leq 6$ that the proposed coding technique and controller is not able to stabilize the unstable dynamic system as the channel input power constraint is violated. But for $7 \leq M \leq N = 10$ that the channel input power constraint is not violated, the stability is achieved using the proposed coding technique and controller, in which as shown in Fig. 3, the difference between J^* and J_{opt} for different $M \in \{7, 8, 9, 10\}$ is very small and as M goes to N this difference becomes smaller. This indicates that when the channel input power constraint is not violated, the proposed coding technique and controller are able to stabilize the system and when the feedback channel is available frequently (and hence the channel input power constraint is not violated), almost nothing is lost in terms of performance. This result is expected as the controller (15) has the same structure as the optimal controller of [10] with the same static controller gain. Also, the recursive equation (10) is exactly the same as the recursive equation of [10] for the decoder output.

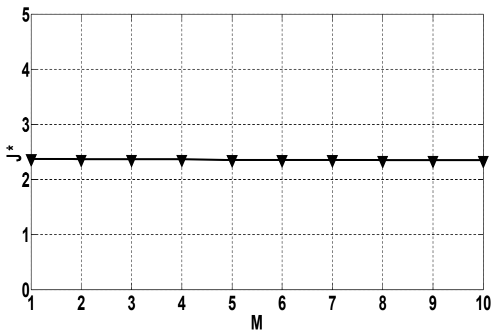

Fig. 2. Trade-off between J^* and M for stable system.

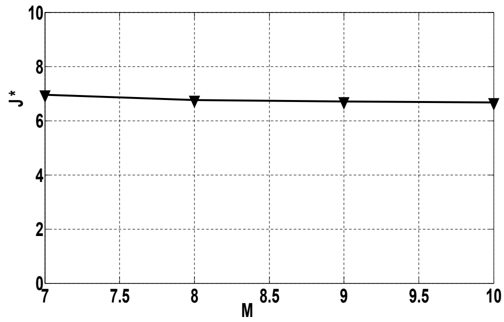

Fig. 3. Trade-off between J^* and M for unstable system.

For unstable system, Fig. 4 illustrates the power of the signal to be transmitted by AWGN channel (i.e., $\alpha_t k_t$) for $M = 1$. As this power violates the channel input power constraint, a distorted version of the signal $\alpha_t k_t$ is received by the decoder. As shown in Fig. 5, as a result of this violation, the proposed coding technique and controller are not able to stabilize the system. Fig. 6 illustrates the power of the signal $\alpha_t k_t$ for $M = 9$. As this power is within the channel input power constraint, the proposed coding technique and controller as shown in Fig. 7 are able to stabilize the system in bounded mean square sense around the origin.

V. CONCLUSION

In this paper when deterministic noiseless feedback channel is available intermittently, for both stable and unstable systems, an innovation encoder, the Kalman filter decoder and a certainty equivalent controller were presented for stability and tracking of linear Gaussian systems over AWGN channel provided channel input power constraint is not violated and the system is controllable. Then, it was illustrated that when feedback channel is frequently available, performance of the system is similar to the optimal performance which corresponds to the case of full time availability of noiseless feedback channel. Therefore, it is concluded that for both stable and unstable linear Gaussian systems, when deterministic feedback channel is available frequently

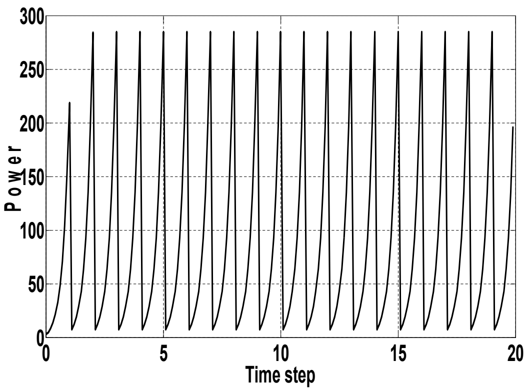

Fig. 4. Power of the signal to be transmitted by AWGN channel for unstable system and $M = 1$.

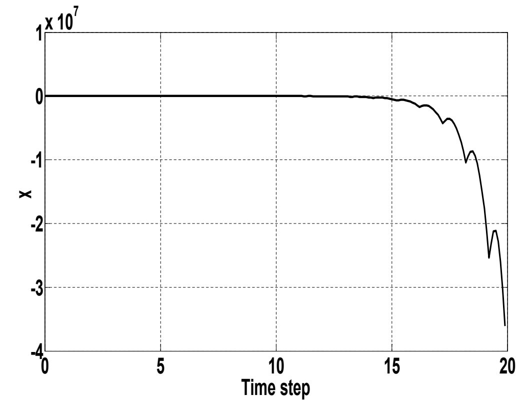

Fig. 5. State trajectory for unstable system and $M = 1$.

and the system is controllable, innovation encoder, the Kalman filter decoder and certainty equivalent controller are good choices for stability and tracking over AWGN channel.

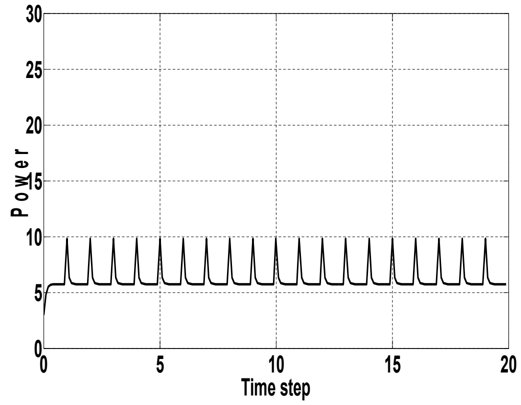

Fig. 6. Power of the signal to be transmitted by AWGN channel for unstable system and $M = 9$.

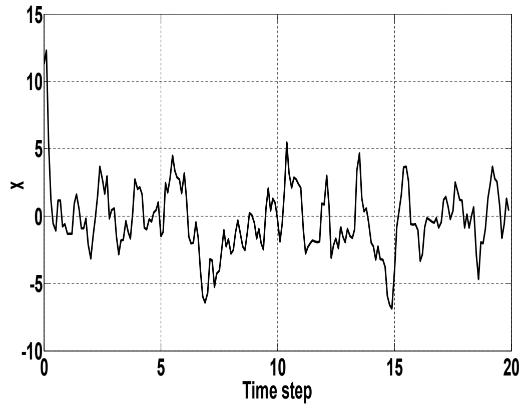

Fig. 7. State trajectory for unstable system and $M = 9$.

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