Feedback Channel In Linear Noiseless Dynamic Systems Controlled Over The Packet Erasure Network

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Abstract

This paper is concerned with tracking state trajectory at remote controller, stability and performance of linear time-invariant noiseless dynamic systems with multiple observations over the packet erasure network subject to random packet dropout and transmission delay that does not necessarily use feedback channel full time. Three cases are considered in this paper: i) without feedback channel, ii) with feedback channel intermittently, and iii) with full time availability of feedback channel. For all three cases, coding strategies that result in reliable tracking of state trajectory at remote controller with asymptotically zero mean absolute estimation error are presented. Asymptotic mean absolute stability of the controlled system equipped with each of these coding strategies is shown; and trade-offs between duty cycle for feedback channel use, transmission delay and performance, which is defined in terms of the settling time, are studied.

Index-Terms: Networked control system, stability and performance, feedback channel.

1 Introduction

1.1 Motivations and Background

Supervisory control systems have been widely used as they enable high level planning, monitoring, and intervention in case of emergency. On the other hand, most of today's embedded systems are wireless enabled due to the vast availability of cheap wireless modems. Therefore, there is an increasing interest in supervisory control systems over wireless communication networks.

Fig. 1 illustrates such a supervisory control system, which will be studied in this paper.

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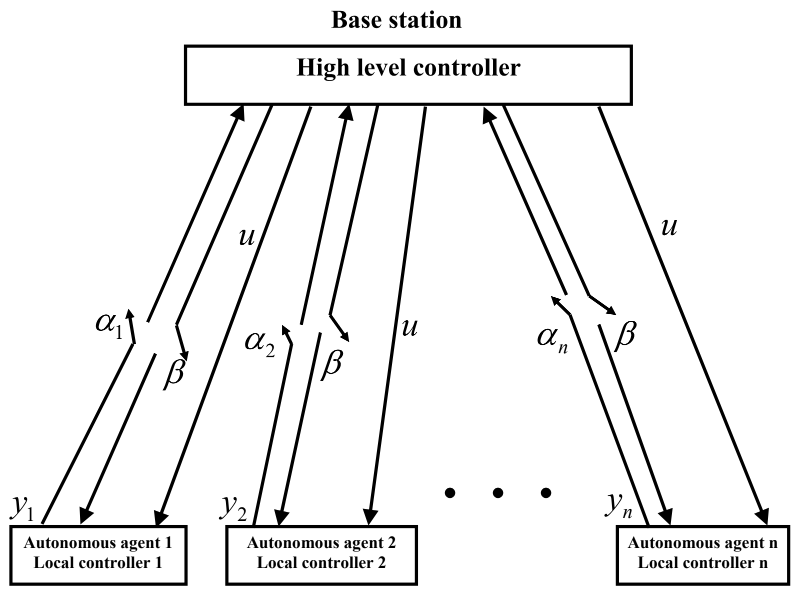

Figure 1: A supervisory control system over the packet erasure network.

This system can represent a fleet of autonomous underwater vehicles (agents) supervised by an autonomous surface vessel (base station); or a fleet of unmanned aerial vehicles supervised by a remote base station. In the control system of Fig. 1, each agent provides measurements for high level controller (supervisory controller) and executes high level commands produced by high level controller. Transmission of information between each agent and high level controller is wireless and, in general, subject to communication imperfections, such as long transmission delay and noise. In the system of Fig. 1, it is assumed that each agent is equipped with limited power supply and hence is limited to transmit with low power. Therefore, the communication from each agent to high level controller is subject to quantization and noise, which results in distortion and packet dropout with random erasure probability α_i. That is, the communication from agents to high level controller is via the packet erasure network. However; as the base station can broadcast with high power, the communication of control signal from high level controller to each agent can be assumed perfect. Therefore, from the point of view of high level controller, the system shown in Fig. 1 can be viewed as a control system with multiple observations when the transmission of observation vector is subject to imperfections.

Thus, the system of Fig. 1 represents a networked control system. Some results addressing the basic problems in stability and estimation of networked control systems can be found in [1]-[17]. [10] addressed the problem of asymptotic mean square estimation of a bounded random variable over the binary erasure channel without feedback channel [1]. In [11] the authors addressed the problem of state estimation of distributed uncontrolled nonlinear Lipschitz systems subject to bounded process and measurement noises over the packet erasure network. The objective in [11] is bounded mean absolute tracking of state trajectory at a remote fusion center when noiseless feedback channel is available full time.[12] addressed the problem of asymptotic stability and tracking of linear noiseless dynamic systems over the digital noiseless channel (and hence when noiseless feedback channel is available full time); and [13] addressed the stability problem of nonlinear noiseless systems over the digital noiseless channel. [14] addressed the problem of asymptotic almost sure stability and tracking of the state trajectory at remote controller of linear noiseless dynamic systems over the packet erasure channel when noiseless feedback channel is available full time; and [15],[16],[17] addressed the problem of mean square stability and tracking of linear Gaussian dynamic systems over Additive White Gaussian Noise (AWGN) channel when noiseless feedback channel is available full time. Similar to the control system of Fig. 1, [12], [13], [14], [15], [16], [17] assumed that the communication of control signal from remote controller to system is perfect.

The above literature review reveals that many results in the literature have been developed under the assumption of full time availability of noiseless feedback channel. Specifically, to the best of our knowledge, there is not any result for stability and tracking with intermittent noiseless feedback channel over the packet erasure channel, which is an important class of digital communication channels as it is an abstract model for the commonly used information technologies, such as the Internet, WiFi and mobile communication. Nevertheless, the availability of noiseless feedback channel may not be possible full time. The full time availability of noiseless feedback channel requires that feedback channel signal is transmitted with high power full time. This results in significant power consumption at the receiver (where the remote controller is located) if the control signal is also transmitted with high power from receiver full time (see Fig. 1). Hence, as transmission of two high power signals full time will result in a short life time for the transmitter of the receiver, this paper aims to relax the full time availability assumption of noiseless feedback channel and address the stability and tracking problem of linear noiseless dynamic systems over the packet erasure network, as shown in Fig. 1, with intermittent noiseless feedback channel.

[1] An Acknowledgment from receiver to transmitter that represents the channel output.

1.2 Paper Contributions

As clear from the above discussion, in the control system of Fig. 1 it is more desirable to use noiseless feedback channel that is available intermittently to avoid exhausting the receiver power supply. Hence, to overcome the deficiency of the available results in the literature, as described above, this paper is concerned with tracking state trajectory at remote controller (high level controller), stability and performance of the supervisory control system of Fig. 1 described by linear time-invariant noiseless subsystems and the packet erasure network when noiseless feedback channel is not necessarily available full time. To model such a feedback channel, in the system of Fig. 1, feedback channel is represented by switches with known duty cycle $\beta \in [0,1]$, where β is a rational number. $\beta = 0$ corresponds to the case of non-availability of feedback channel while $\beta = 1$ corresponds to its full time availability. In the system of Fig. 1 performance is described by the settling time. This paper overcomes the deficiency of the available results in the literature by presenting coding strategies and controller that guarantee asymptotic mean absolute tracking of state trajectory at remote controller and stability for the following three cases: i) without feedback channel (i.e., $\beta = 0$), ii) with feedback channel intermittently (i.e.,$\beta \in (0,1)$), and iii) with full time availability of feedback channel ($\beta = 1$). Asymptotic mean absolute stability of the system of Fig. 1 equipped with each of these coding strategies is shown. Trade-offs between duty cycle for feedback channel use, transmission delay and performance are also studied in this paper.

1.3 Paper Organization

The paper is organized as follows. Section 2 describes the supervisory control system of Fig. 1 in more detail. Section 3 presents coding strategies that compensate the effects of distortion and random packet dropout and provides reliable tracking. This is followed by the stability and performance analysis in Section 4. In Section 5, the paper is concluded by summarizing the main contributions of the paper.

2 Problem Formulation

In this paper, the following conventions are used. $z(t)$ denotes the value of the signal z at time $t \geq 0$. $z[m]$, where m is a non-negative integer, denotes the value of the signal z at sampling instant m. " $\doteq$ " denotes "by definition equals", $\mathbf{N}_+ \doteq \{0, 1, 2, ...\}$ and $A^{'}$ denotes the transpose of a vector/matrix A. $\lambda_l(M)$ denotes the lth eigenvalue of the square matrix M, the block diagonal matrix is denoted by $diag(\cdot)$, and $\sigma_{max}(B)$ denotes the largest singular value of matrix B. $\log(\cdot)$ denotes the logarithm of base 2, $\lfloor a \rfloor$ denotes the floor of scalar a

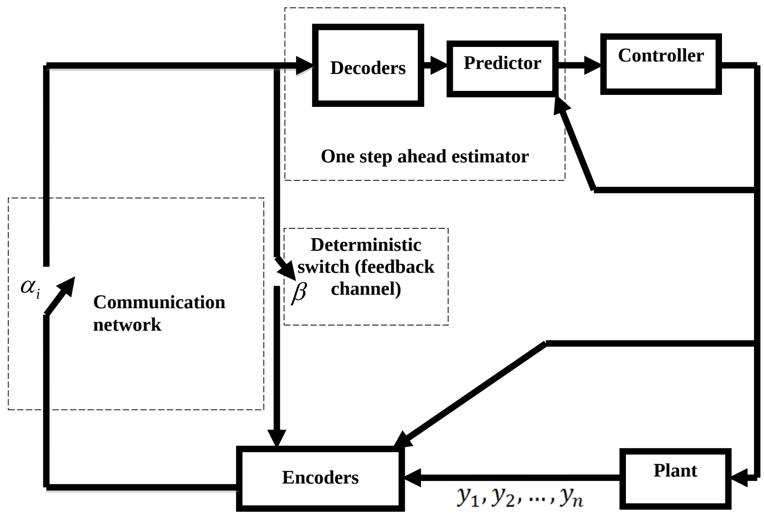

Figure 2: An unstable dynamic system over the packet erasure network.

and $E[\cdot]$ denotes the expected value. Euclidean norm is denoted by $||\cdot||$, the absolute value by $|\cdot|$, the set of real numbers by $\Re$ and the standard uniform distribution is denoted by $U(0,1)$. $\prod_{j=1}^{k} a_j$ denotes the ordered product, i.e., $\prod_{j=1}^{k} a_j \doteq a_k.a_{k-1}.....a_1$, and $\mathbf{1}$ stands for the indicator function.

2.1 Description of the System

This paper is concerned with tracking state trajectory, stability and performance of the supervisory control system of Fig. 1 described by Linear Time-Invariant (LTI) subsystems over the packet erasure network, as shown in more detail in Fig. 2. In the block diagram of Fig. 2, we denote by y_i the value of the ith sampled data (the measurement vector of the ith subsystem) at time instant k, when the control action is updated. To avoid collision of transmitted sampled data at controller, sampled data: $y_1, y_2, ..., y_n$, which are all sampled at time instant k, are transmitted sequentially, like a Time Division Multiple Access (TDMA) scheme, in which, the time period between transmission of two successive samples is a positive scalar T_m. The time period T_m is referred as sequential transmission period. The time period between two successive time instants k and $k+1$ is $T_\beta(> T_m)$. The control action is updated with the time period T_β; and measurements are provided from subsystems with the same time period (see Fig. 3). During each time period that the control action is updated, feedback channel can be used. That is, an acknowledgment bit is sent from receiver of controller to transmitter

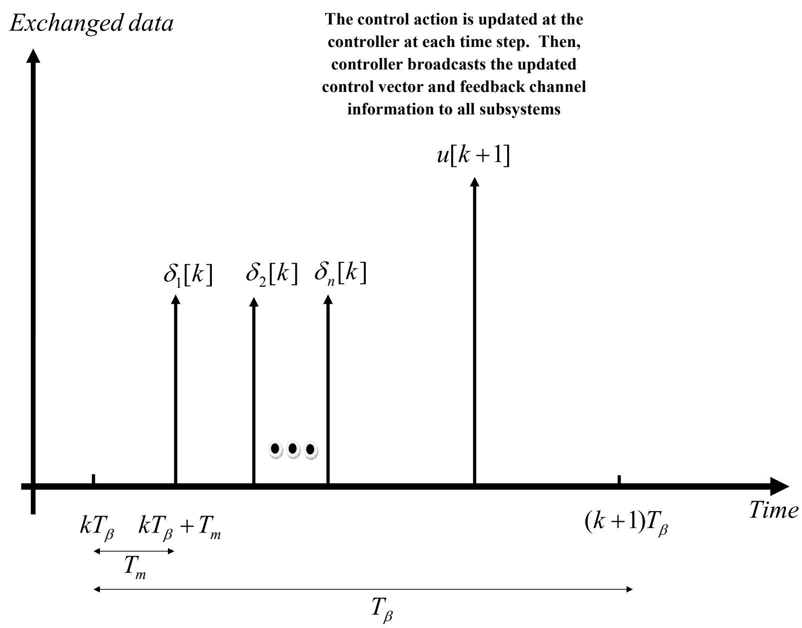

Figure 3: A TDMA scheme for exchanging information between the plant and controller.

of each subsystem to indicate whether a transmission from a subsystem has been successful or not. This feedback channel is shown in Fig. 2 by a switch with known on/off duty cycle β.

The building blocks of Fig. 2 are described now.

Plant: Plant consists of the following linear time-invariant noiseless subsystems:

$$\begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_i u(t), \\ y_i(t) = C_i x_i(t), \qquad i = \{1, 2, ..., n\}, \end{cases} \tag{1}$$

where $A_i \in \Re^{n_i \times n_i}$, $B_i \in \Re^{n_i \times m}$, $C_i \in \Re^{l_i \times n_i}$, $x_i \in \Re^{n_i}$, $u \in \Re^m$, $y_i \in \Re^{l_i}$, and $k \in \mathbf{N}_+$. The initial states $x_i(0)$, $i \in \{1, 2, ..., n\}$, are Random Variables (RVs) with bounded supports. That is, for each i, there exists a closed bounded set $\Gamma_i \subset \Re^{n_i}$ such that $x_i(0) \in \Gamma_i$.

In the problem considered in this paper the control update action is of the hold type. That is, at each time instant k, $u(kT)$ is applied and held until the next time instant. Also, at each time instant k, y_is are sampled. Therefore, the plant has the following discrete-time equivalent representation.

$$\begin{cases} x_i[k+1] = A_{di}(\beta) x_i[k] + B_{di}(\beta) u[k], \\ y_i[k] = C_i x_i[k], \ x_i[0] \doteq x_i(0), \qquad i = \{1, 2, ..., n\}, \end{cases} \tag{2}$$

where $A_{di}(\beta) = e^{A_i T_\beta}$, $B_{di}(\beta) = (\int_0^{T_\beta} e^{A_i \tau} d\tau) B_i$. Throughout, it is assumed that there exists a matrix K_β such that the matrix $A_d(\beta) + B_d(\beta) K_\beta$ is stable, where

$$A_d(\beta) \doteq diag(A_{d1}(\beta), ..., A_{dn}(\beta)), \ B_d \doteq diag(B_{d1}(\beta), ..., B_{dn}(\beta)).$$

Communication network: The communication channel from each subsystem to controller is the packet erasure channel. It is a digital channel consisting of modulator, noisy media, and de-modulator, which transmits a packet of binary data (subject to transmission delay) in each channel use. Let $\delta_i(kT_\beta)$ denote the channel input, which is a packet of binary data that includes information bits as well as overhead bits added by channel encoder for error detection and correction. Let, also $\bar{\delta}_i(kT_\beta + d_{ik})$ denote the corresponding channel output, where d_{ik} is the unknown time varying transmission delay. Also, let e denote the erasure symbol. Then,

$$\bar{\delta}_i(kT_\beta + d_{ik}) = \mathcal{C}(\delta_i(kT_\beta)) \doteq \begin{cases} \delta_i(kT_\beta) & \text{with probability } 1-\alpha_i \\ e & \text{with probability } \alpha_i \end{cases}$$

That is, this channel erases a transmitted packet with probability α_i whenever channel decoder detects flipped bits (due to transmission noise) that cannot be corrected by the implemented error correction technique.
Note that the binary erasure channel is a special case of the packet erasure channel that transmits one information bit in each channel use.

Throughout, it is assumed that the erasure probability α_i is known a priori. Also, the upper and lower bounds on unknown time varying transmission delay are known (i.e., $\underline{T}_{di} \leq d_{ik} \leq \overline{T}_{di}$, where $\underline{T}_{di}$ and $\overline{T}_{di}$ are known). Moreover, the input of modulator is binary; while the output of de-modulator is ternary with three states: "0", "1", and "Idle" such that when the channel is not in use, de-modulator outputs the "Idle" state.

Deterministic switch: During each time period that the control action is updated, a feedback channel can be used from receiver of controller to transmitter of each subsystem. That is, a noiseless acknowledgment bit from controller to each subsystem that indicates whether a transmission from a subsystem has been successful or not. This is modeled by a switch with a known switching policy to all transmitters and receivers, in which duty cycle for turning on this switch (i.e., using feedback channel) is $\beta = \frac{q}{p}$, $q \in \mathbf{N}_+$, $p \in \{1, 2, 3, ...\}$, $q \leq p$. $\beta = 0$ corresponds to the case of non-availability of feedback channel and for the case of $\beta \neq 0$, in each p updates of the control action, in the first q updates feedback channels are used. It is assumed that in the first time step (i.e., within the time period between the time instants $k = 0$ and $k = 1$) feedback channel is used.

In the closed loop feedback system of Fig. 2, encoders and decoders are used to compensate the effects of random packet dropout. As the effect of channel encoder and channel decoder in improving the transmission quality over a noisy media is present in the channel model, without loss of generality, in what follows we only focus on source encoding and source decoding operations.

Having that, encoders and decoders are described as follows.

Encoders: There is an encoder for each subsystem, which is a causal operator denoted by $\mathcal{E}_{i\beta}(\cdot)$, $i = \{1, 2, ..., n\}$. For each subsystem i, let the random variable σ_{ik} denote the computational latency associated with the ith encoding operation during time instants k and $k+1$. Within this time period, each encoder $\mathcal{E}_{i\beta}(\cdot)$ maps the corresponding subsystem output: $y_i(kT_\beta + (i-1)T_m)$ to the channel input: $\delta_i(kT_\beta + (i-1)T_m + \sigma_{ik})$, which is a string of binaries with length $\mathcal{R}_{ik}$. In performing the above operation, the information about the status of the previous transmission (its success or fail) is used in the encoding operation if the deterministic switch was on at the previous time step. The encoders can also use the sequence of control inputs up to time instant k in performing the above task.

Decoders: Decoders are also causal operators, which are denoted by $\mathcal{D}_{i\beta}(\cdot)$, $i = \{1, 2, ..., n\}$. They map the channel outputs $\bar{\delta}_i(kT_\beta + (i-1)T_m + \sigma_{ik} + \frac{\mathcal{R}_{ik}}{BW} + d_{ik})$ to the state estimates $\hat{x}_i(kT_\beta + (i-1)T_m + \sigma_{ik} + \frac{\mathcal{R}_{ik}}{BW} + d_{ik} + \gamma_{ik})$, $i = 1, 2, ..., n$, where BW is the bit rate of the communication link, the random variable γ_{ik} is the computational latency due to decoding operation and $\hat{x}_i(kT_\beta + (i-1)T_m + \sigma_{ik} + \frac{\mathcal{R}_{ik}}{BW} + d_{ik} + \gamma_{ik})$, which is abbreviated by $\hat{x}_i[k|k]$ for the simplicity of presentation, is the estimation of $x_i[k]$. In performing the decoding operation for time instant k, the decoders can use the sequence of control inputs up to time instant $k-1$.

Controller: Control update action is of the hold type. That is, whenever the control action is updated, the control value is applied on the plant and held until the next control value is updated. In this paper controller has the following representation:

$$u[0] = 0, \quad u[k+1] = K_\beta \hat{x}[k+1], \tag{3}$$

where

$$\hat{x}[k+1] \doteq \begin{pmatrix} \hat{x}_1[k+1] \\ . \\ . \\ . \\ \hat{x}_n[k+1] \end{pmatrix} = A_d(\beta) \begin{pmatrix} \hat{x}_1[k|k] \\ . \\ . \\ . \\ \hat{x}_n[k|k] \end{pmatrix} + B_d(\beta)u[k]$$

represents the operation of the predictor block of the one step ahead estimator block of Fig. 2. In (3), $\hat{x}[k+1]$ is the output of the one step ahead estimator block, $\hat{x}_i[k|k]$ are the outputs of the decoders, and the controller gain $K_\beta \in \Re^{m\times(n_1+...+n_n)}$ is defined such that asymptotic mean absolute stability, as defined in the following, is achieved.

2.2 Definitions and Objectives

Definition 2.1 (**Asymptotic Mean Absolute Stability**)*: The linear time-invariant noiseless dynamic system (2) in the closed loop feedback system of Fig. 2 is asymptotically mean absolute stable if there exist a β, encoding policies $\mathcal{E}_{i\beta}(\cdot)$, decoding policies $\mathcal{D}_{i\beta}(\cdot)$, and a controller gain K_β such that the following property holds for all choices of the initial state*

$$\lim_{k\to\infty} E||x[k]|| = 0, \quad x[k] \doteq \begin{pmatrix} x_1[k] \\ . \\ . \\ . \\ x_n[k] \end{pmatrix}.$$

In addition of the asymptotic mean absolute stability, in this paper we are interested in asymptotic mean absolute tracking of state trajectory at controller. This objective is defined in the following

Definition 2.2 (**Asymptotic Mean Absolute Tracking**)*: The linear time-invariant noiseless dynamic system (2) in the closed loop feedback system of Fig. 2 is asymptotically mean absolute tractable at the controller if there exist a β, encoding policies $\mathcal{E}_{i\beta}(\cdot)$ and decoding policies $\mathcal{D}_{i\beta}(\cdot)$ such that the following property holds for all choices of the initial state*

$$\lim_{k\to\infty} E||x[k] - \hat{x}[k]|| = 0.$$

Here, $\hat{x}[k]$ is the estimation of the state vector of the system at controller.

Next, coding computational latency is defined which is used to define the sequential transmission period.

Definition 2.3 (Coding Computational Latency): *Throughout, the following quantities, measured in second,*

$$\mathcal{C}_c(\beta) \doteq \sup_{i\in\{1,2,\ldots,n\},k\in\mathbf{N}_+} \sigma_{ik} \quad \textit{and} \quad \mathcal{C}_d(\beta) \doteq \sup_{i\in\{1,2,\ldots,n\},k\in\mathbf{N}_+} \gamma_{ik}$$

are referred as encoding and decoding computational latencies, respectively.

To avoid coding computation overflow and collision of the sequentially transmitted packets of data due to different transmission delays, throughout it is assumed that

$$T_m \geq \mathcal{C}_c(\beta) + T_c, \quad T_c \doteq \max_{i\in\{1,2,\ldots,n\},k\in\mathbf{N}_+} \{\overline{T}_{di} + \frac{\mathcal{R}_{ik}}{BW}\} - \min_{i\in\{1,2,\ldots,n\},k\in\mathbf{N}_+} \{\underline{T}_{di} + \frac{\mathcal{R}_{ik}}{BW}\}, \qquad (4)$$

$$T_m + \min_{i\in\{1,2,...,n\},k\in\mathbf{N}_+} \{\underline{T}_{di} + \frac{\mathcal{R}_{ik}}{BW}\} \geq \mathcal{C}_d(\beta). \tag{5}$$

Conditions (4) and (5) give enough time to encoders and decoders, respectively, to perform their actions without encountering computation overflow.

To also give enough time to the controller to update its action, the controller sampling period T_β is chosen as follows ($\mathcal{R}_u$ is the length of binary representation of control signal).

$$\begin{aligned} T_\beta &\doteq nT_m + \max\{T_{max1}, \mathcal{C}_d(\beta)\} + \frac{\mathcal{R}_u + n}{BW} + T_{max2}, \\ T_{max1} &\doteq \max_{i\in\{1,2,...,n\},k\in\mathbf{N}_+} \{\overline{T}_{di} + \frac{\mathcal{R}_{ik}}{BW}\},\ T_{max2} \doteq \max_{i\in\{1,2,...,n\}} \{\overline{T}_{di}\}. \end{aligned} \tag{6}$$

Next, we define the settling time, which indicates the performance of the closed loop system of Fig. 2.

Definition 2.4 *(***The Settling Time***): Throughout, the smallest time $T_s^\epsilon(\beta)$, under which for a given $\epsilon > 0$ the following inequality holds for all choices of the initial states, is referred as the settling time.*

$$E||x(t)|| \leq \epsilon, \quad \forall t \geq T_s^\epsilon(\beta), \quad x(t) \doteq \begin{pmatrix} x_1(t) \\ . \\ . \\ . \\ x_n(t) \end{pmatrix}.$$

The main objective of this paper is to address the problem of asymptotic mean absolute stability and tracking of the control system of Fig. 2 when feedback channel is not necessarily used full time. Other objective is to study trade-offs between performance, transmission delay and duty cycle for feedback channel use. To address these questions, in the next sections we present three types of coding strategies: i) strategy with no feedback channel, ii) strategy with intermittent feedback channel, and iii) strategy with full time availability of feedback channel. These strategies result in asymptotic mean absolute tracking. Subsequently, asymptotic mean absolute stability of the system of Fig. 2 equipped with each of these strategies is shown. Then, we discuss trade-offs between duty cycle for feedback channel use, transmission delay and performance.

3 Coding Design

For large erasure probabilities α_i, $i \in \{1, 2, ..., n\}$, the stability of the system of Fig. 2 may not be possible without compensating the effects of data dropout. Therefore, in this section we propose coding strategies, which compensate the effects of communication error and provide asymptotic mean absolute tracking of state trajectory at controller. We present such coding strategies for three cases: i) $\beta = 0$ (non-availability of feedback channel), ii) $\beta = \frac{q}{p}$, $q, p \in \{1, 2, 3, ...\}$, $q < p$ (feedback channel intermittently), and iii) $\beta = 1$ (full time availability of feedback channel).

Note that following the assumption made on the controller sampling period and as the control action is updated based on one step ahead prediction (see Fig. 2), the design of coding strategies can be done independent of the computational latencies, transmission delays, and delays induced by TDMA scheme and limited communication bit rate, i.e., without loss of generality, in this section it is assumed that $\sigma_{ik} = \gamma_{ik} = d_{ik} = 0$.

3.1 Coding Strategy - $\beta = 0$

For $\beta = 0$ (i.e., non-availability of feedback channel), a coding strategy is presented, which is based on the coding strategy of [10]. The coding strategy of [10] estimates initial states of the system (2) using an anytime coding strategy that does not use feedback channel. The details of this coding strategy are given in the Appendix. From ([10], Theorem 6.1) it follows that using this strategy, the initial state of the system (2) are estimated in mean square sense, in which the estimation error decreases, as time increases. That is, for each $i \in \{1, 2, ..., n\}$, the following inequality holds.

$$E||x_i[0] - \hat{x}_i[0|k]||^2 \leq c_i^2 k 2^{-2\Delta(R_i, n_i, \alpha_i)k},$$

where $\hat{x}_i[0|k]$ is the estimation of the initial state $x_i[0]$ at the end of the decoder of Fig. 2 at time instant k, $c_i > 0$ is a constant depending only on α_i and R_i (which is related to the information transmission rate as follows: $\mathcal{R}_{ik} = \lfloor R_i.(k+1) \rfloor$), and $\Delta(R_i, n_i, \alpha_i) \doteq \min\{\frac{R_i}{n_i}, \frac{1}{2} \min_{0 \leq \eta_i \leq 1} H(\eta_i || 1 - \alpha_i) + [\eta_i - R_i]_+\}$, where $H(x||y) \doteq x \log \frac{x}{y} + (1-x) \log \frac{1-x}{1-y}$ and $[x]_+ \doteq \max\{0, x\}$.

After estimating the initial state of the system (2) over the packet erasure channels using the coding strategy of [10], the output of the one step ahead estimator block of Fig. 2 is updated as follows.

$$\hat{x}[k+1] = A_d(\beta)\hat{x}[k|k] + B_d(\beta)u[k],$$

where

$$\hat{x}[k|k] \doteq A_d^k(\beta)\hat{x}[0|k] + \sum_{j=0}^{k-1} (A_d(\beta))^{k-1-j} B_d(\beta)u[j], \;\; (\hat{x}[0|k] \doteq \begin{pmatrix} \hat{x}_1[0|k] \\ . \\ . \\ . \\ \hat{x}_n[0|k] \end{pmatrix}). \tag{7}$$

In the following proposition, it is shown that under some conditions, asymptotic mean square tracking of the state trajectory of the system (2) over the packet erasure channels is possible using the coding strategy of [10]. That is,

$$E||x[k]-\hat{x}[k|k]||^2 \to 0, \text{ as } k \to \infty.$$

Proposition 3.1 *Suppose that* $\Delta(R_i, n_i, \alpha_i) > 2\log\sigma_{max}(A_{di}(\beta))$ *for each* $i \in \{1, 2, ..., n\}$, *where* $\sigma_{max}(A_{di}(\beta))$ *is the largest singular value of the matrix* $A_{di}(\beta)$. *Then, if we use the coding strategy of [10] to estimate the initial state of the system (2) at the end of the decoders, we have asymptotic mean square tracking over the packet erasure channels with erasure probabilities* α_i, $i \in \{1, 2, ..., n\}$.

Proof: Using the coding strategy of [10] the following inequality holds.

$$\begin{aligned} E||x[k]-\hat{x}[k|k]||^2 &= \sum_{i=1}^{n} E||(A_{id}(\beta))^k(x_i[0]-\hat{x}_i[0|k])||^2 \\ &\leq \sum_{i=1}^{n}(\sigma_{max}(A_{di}(\beta))^{2k} c_i^2 k 2^{-\Delta(R_i,n_i,\alpha_i)k} \\ &= \sum_{i=1}^{n} c_i^2 \frac{k}{2^{(\Delta(R_i,n_i,\alpha_i)-2\log\sigma_{max}(A_{di}(\beta))k}}. \end{aligned} \tag{8}$$

Now, by applying the rule of Hopital - Bernoulli for limits, we have:

$$\lim_{k\to\infty} \frac{k}{2^{(\Delta(R_i,n_i,\alpha_i)-2\log\sigma_{max}(A_{di}(\beta)))k}} = \lim_{k\to\infty} \frac{1}{\gamma_i \ln 2.2^{\gamma_i k}} = 0,$$

where $\gamma_i \doteq \Delta(R_i, n_i, \alpha_i) - 2\log\sigma_{max}(A_{di}(\beta))$. That is, under the assumption of $\Delta(R_i, n_i, \alpha_i) > 2\log\sigma_{max}(A_{di}(\beta))$ for each $i \in \{1, 2, ..., n\}$, the right hand side of (8); and hence, $E||x[k]-\hat{x}[k|k]||^2$ converge to zero, as $k \to \infty$. This completes the proof.

We have the following remarks and corollary following the above result.

Remark 3.2 *i) From the hierarchy of moment convergence [18], it follows that asymptotic mean square tracking implies asymptotic mean absolute tracking. That is,* $E||x[k]-\hat{x}[k|k]|| \to 0$, *as* $k \to \infty$.
ii) From the Holder inequality [18], it follows that an upper bound on $E||x[k]-\hat{x}[k|k]||$ *is given by*

$$E||x[k]-\hat{x}[k|k]|| \leq \sqrt{E||x[k]-\hat{x}[k|k]||^2} \leq \sqrt{\sum_{i=1}^{n} c_i^2 \frac{k}{2^{(\Delta(R_i,n_i,\alpha_i)-2\log\sigma_{max}(A_{di}(\beta)))k}}}. \tag{9}$$

Remark 3.3 *i) Among the available coding strategies that do not use feedback channel and provide reliable tracking of the state trajectory of dynamic systems via estimating the initial state, the coding strategy of [10] has the fastest decoding error decay rate (it has an exponential decay rate). However, the decay rate of this strategy is not as fast as the strategies that use feedback channel. Consequently, for unstable system matrix $A_d(\beta)$, the coding strategy (7) may not guarantee reliable tracking.*
ii) From [10] it follows that for each i the number of required binary operations for generating each source codeword follows from a binomial distribution with parameter $\mathcal{R}_{ik}$ and $\frac{1}{2}$. On the other hand, reconstructing the initial state at the receiver requires at least $\mathcal{O}(k^2)$ and at most $\mathcal{O}(k^3)$ binary operations for each time instant k. Therefore, the coding strategy (7) is computationally expensive.

Corollary 3.4 *Let $e[k] \doteq x[k] - \hat{x}[k]$ be the estimation error and $\hat{x}[k]$ the output of the one step ahead estimator block of Fig. 2, which is updated as follows:*

$$\hat{x}[k+1] = A_d(\beta)\hat{x}[k|k] + B_d(\beta)u[k].$$

Then,

$$\begin{aligned} e[k] &= x[k] - \hat{x}[k] \\ &= A_d(\beta)(x[k-1] - \hat{x}[k-1|k-1]) \\ &\Rightarrow \\ ||e[k]|| &= \sigma_{max}(A_d(\beta))||x[k-1] - \hat{x}[k-1|k-1]||. \end{aligned}$$

Consequently, from Remark 3.2, it follows that

$$E||e[k]|| \leq \sigma_{max}(A_d(\beta))\sqrt{\sum_{i=1}^{n} c_i^2 \frac{k-1}{2^{(\Delta(R_i,n_i,\alpha_i)-2\log\sigma_{max}(A_{di}(\beta)))(k-1)}}}, \tag{10}$$

and therefore $E||e[k]||$ converges to zero as $k \to \infty$ if the following condition holds:

$$\Delta(R_i, n_i, \alpha_i) > 2\log\sigma_{max}(A_{di}(\beta)), \quad i = \{1, 2, ..., n\}.$$

3.2 Coding Strategy - $\beta = \frac{q}{p}$, where $p, q \in \{1, 2, 3, ...\}$, $q < p$

In this section, for $\beta = \frac{q}{p}$, $p, q \in \{1, 2, 3, ...\}$, $q \leq p$, we develop a differential coding strategy that uses feedback channel intermittently. That is, in each p updates of the control action, in the first q updates, feedback channels are used in all encoders $\mathcal{E}_{i\beta}(\cdot)$, $i = 1, 2, ..., n$. For the simplicity of presentation, without loss of generality, in what follows it is assumed that the matrices C_i, $i \in \{1, 2, ..., n\}$ in (2) are identity matrices (if a matrix C_i is not an identity matrix but $C_i'C_i$ is invertible, then $\bar{y}_i \doteq (C_i'C_i)^{-1}C_i'y_i = x_i$ is treated as the observation signal). It is also assumed that for each i, the encoder $\mathcal{E}_{i\beta}(\cdot)$ and the corresponding decoder $\mathcal{D}_{i\beta}(\cdot)$ are aware of each other policies.

In addition of the above assumptions, for the simplicity of presenting the encoding and decoding operations, in what follows, without loss of generality, it is assumed that subsystems are scalar. Having these assumptions, at time instant $k \in \mathbf{N}_+$, the ith encoder $\mathcal{E}_{i\beta}(\cdot)$ first computes the error $x_i[k] - \hat{x}_i^e[k]$, where $\hat{x}_i^e[k]$ is the encoder estimate from the corresponding decoder output, as defined as follows.
For $k = 0$,

$$\hat{x}_i^e[k] = 0,$$

and for $k \neq 0$, if the feedback channel is used during the time period between time instants $k-1$ and k, then as the ith encoder is aware of the policy of the ith decoder, by the knowledge of the feedback acknowledgment it can determine $\hat{x}_i[k-1|k-1]$; and subsequently, it updates its estimation from the decoder output as follows:

$$\hat{x}_i^e[k] = A_{di}(\beta)\hat{x}_i[k-1|k-1] + B_{di}(\beta)u[k-1].$$

Otherwise,

$$\hat{x}_i^e[k] = A_{di}(\beta)\hat{x}_i^e[k-1] + B_{di}(\beta)u[k-1].$$

Next, the encoder $\mathcal{E}_{i\beta}(\cdot)$ partitions the interval $[-L_i[k], L_i[k]]$, where $x_i[k] - \hat{x}_i^e[k]$ lies in (i.e., $|x_i[k] - \hat{x}_i^e[k]| \leq L_i[k]$) into $2^{\mathcal{R}_i}$ equal size bins. Note that the ith encoder and decoder can determine the upper bound $L_i[k]$ using feedback channel (if available) and the knowledge of each other policies. Subsequently, the center of each bin is chosen as the index of the bin, which is represented by $z \in \{0, 1, ..., 2^{\mathcal{R}_i} - 1\}$, where $z = 0$ corresponds to the center of the first bin: $[-L_i[k], -(1 - 2^{1-\mathcal{R}_i})L_i[k])$, which is $l_{i0}[k] = -(1 - 2^{-\mathcal{R}_i})L_i[k]$, $z = 1$ corresponds to the center of the second bin: $[-(1 - 2^{1-\mathcal{R}_i})L_i[k], -(1 - 2^{2-\mathcal{R}_i})L_i[k])$, which is $l_{i1}[k] = -(1 - 3.2^{-\mathcal{R}_i})L_i[k]$, and so on and so forth.
Having that, upon observing $x_i[k]$, the ith encoder represents the index of the bin, where $x_i[k] - \hat{x}_i^e[k]$ is located into $\mathcal{R}_i$ information bits and transmits the corresponding packet. The output of the ith decoder $\mathcal{D}_{i\beta}(\cdot)$, i.e., $\hat{x}_i[k|k]$, is then updated as follows:

$$\hat{x}_i[k|k] = \begin{cases} l_{iz}[k] + \hat{x}_i^e[k] & \text{if erasure does not occur} \\ \hat{x}_i^e[k] & \text{if erasure occurs} \end{cases} \tag{11}$$

In (11), $l_{iz}[k]$ is the center of the $z + 1$ bin, which contains $x_i[k] - \hat{x}_i^e[k]$. Note that $\hat{x}_i^e[0] = 0$ and as the encoder $\mathcal{E}_{i\beta}(\cdot)$ and the decoder $\mathcal{D}_{i\beta}(\cdot)$ are aware of each other policies, the decoder can determine $\hat{x}_i^e[k]$. For the vector case, the ith encoder encodes the hth element ($h = \{1, 2, ..., n_i\}$) of the vector $x_i[k] - \hat{x}_i^e[k]$ into $\mathcal{R}_{ih}$ information bits and transmits a packet with length $\mathcal{R}_i = \sum_{h=1}^{n_i} \mathcal{R}_{ih}$ containing information about vector $x_i[k] - \hat{x}_i^e[k]$ over the packet erasure channel.

Now, we have the following proposition, which is used to show asymptotic mean absolute tracking of the system (2) over the packet erasure channels using the above coding strategy.

Proposition 3.5 *Consider the encoding-decoding pairs $(\mathcal{E}_{i\beta}(\cdot), \mathcal{D}_{i\beta}(\cdot))$, as described above. For a given duty cycle $\beta = \frac{q}{p}$, where $p, q \in \{1, 2, 3, ...\}$ $q \leq p$, using these pairs, asymptotic mean square tracking of the form $\lim_{k\to\infty} E||x[k] - \hat{x}[k|k]||^2 = 0$ over the packet erasure channels with erasure probabilities α_i, $i \in \{1, 2, ..., n\}$, is achieved if for each i there exist information rates $\mathcal{R}_{ih}$, $h = \{1, 2, ..., n_i\}$, such that the following inequalities hold*

$$|\lambda_h(A_{di}(\beta))|^{p+q}((1 - \alpha_i)\frac{1}{2^{2\mathcal{R}_{ih}}} + \alpha_i)^q < 1, \quad \forall i, h \tag{12}$$

where $\lambda_h(A_{di}(\beta))$ denote the hth eigenvalue of the matrix $A_{di}(\beta)$.

Proof: For the simplicity of presentation in what follows it is assumed that subsystems are scalar. The extension of the results to the general vector case is straightforward and follows by implementing a similarity transformation that turns matrix $A_{di}(\beta)$ to the real Jordan form.

Having that, as the initial state is bounded, the following inequality holds

$$|x_i[0]| \leq L_i[0],$$

where for each i, $L_i[0]$ is known a priori.

At time instant $k = 0$, upon observing $x_i[0]$, the ith encoder $\mathcal{E}_{i\beta}(\cdot)$ partitions the interval $[-L_i[0], L_i[0]]$ and it determines the bin, where $x_i[0] - \hat{x}_i^e[0]$ ($\hat{x}_i^e[0] = 0$) is located and represents the corresponding index by $\mathcal{R}_i$ bits and transmits the corresponding packet. Then, the output of the ith decoder is updated by (11) (recall that $\hat{x}_i^e[0] = 0$). Consequently, for time instant $k = 0$, the decoding error is bounded above by

$$|x_i[0] - \hat{x}_i[0|0]| \leq V_i[0],$$

where

$$V_i[0] = \begin{cases} \frac{L_i[0]}{2^{\mathcal{R}_i}} & \text{if erasure does not occur} \\ L_i[0] & \text{if erasure occurs} \end{cases}$$

At time instant $k = 1$, using feedback acknowledgment, the encoder can determine $V_i[0]$ and $\hat{x}_i[0|0]$. Subsequently, it computes $\hat{x}_i^e[1] = A_{di}(\beta)\hat{x}_i[0|0] + B_{di}(\beta)u[0]$, and $L_i[1] = |A_{di}(\beta)|V_i[0]$

(note that during the time period between the time instants $k = 0$ and $k = 1$, feedback channel is used). Then, it partitions the interval $[-L_i[1], L_i[1]]$ into $2^{\mathcal{R}_i}$ bins, as described above. Upon observing $x_i[1]$, the encoder computes $x_i[1] - \hat{x}_i^e[1]$ and determines the bin, where $x_i[1] - \hat{x}_i^e[1]$ is located. Then, it represents the index of this bin by $\mathcal{R}_i$ bits and transmits the corresponding packet. Subsequently, the decoder output is updated by (11). Hence, for this case the decoding error is bounded above by

$$|x_i[1] - \hat{x}_i[1|1]| \leq V_i[1],$$

where

$$V_i[1] = \begin{cases} \frac{L_i[1]}{2^{\mathcal{R}_i}} & \text{if erasure does not occur} \\ L_i[1] & \text{if erasure occurs} \end{cases}$$

Consequently, by following the above procedure, the following relationships hold:
At time instant $k \in \{1, 2, ..., q\}$, $L_i[k] = |A_{di}(\beta)|V_i[k-1]$ and

$$|x_i[k] - \hat{x}_i[k|k]| \leq V_i[k],$$

where

$$V_i[k] = \begin{cases} \frac{L_i[k]}{2^{\mathcal{R}_i}} & \text{if erasure does not occur} \\ L_i[k] & \text{if erasure occurs} \end{cases}$$

At time instant $k \in \{q+1, ..., p\}$, $L_i[k] = |(A_{di}(\beta))^{k-q+1}|V_i[q-1]$ and

$$|x_i[k] - \hat{x}_i[k|k]| \leq V_i[k],$$

where

$$V_i[k] = \begin{cases} \frac{L_i[k]}{2^{\mathcal{R}_i}} & \text{if erasure does not occur} \\ L_i[k] & \text{if erasure occurs} \end{cases}$$

And, in general, at time instant k:

$$|x_i[k] - \hat{x}_i[k|k]| \leq V_i[k],$$

where

- for $k = pj, \ \ j \in \{0, 1, 2, 3, ...\}$, we have
  $k = 0$

  $$V_i[0] = F_i[0]L_i[0];$$

  otherwise

  $$V_i[pj] = F_i[pj]|(|A_{di}^{p-q+1}(\beta)|V_i[p(j-1) + q - 1]),$$

  where the sequence $F_i[t]$, $t \in \mathbf{N}_+$, is i.i.d. with the following common distribution

  $$F_i[t] = \begin{cases} \frac{1}{2^{\mathcal{R}_i}} & \Pr(F_i[t] = \frac{1}{2^{\mathcal{R}_i}}) = 1 - \alpha_i \\ 1 & \Pr(F_i[t] = 1) = \alpha_i \end{cases}$$

- for $k \in \{pj+1, ..., pj+q\}$, we have

$$V_i[k] = F_i[k]|A_{di}(\beta)|V_i[k-1],$$

and

- for $k \in \{pj+q+1, ..., pj+p-1\}$, we have

$$V_i[k] = F_i[k]|(|A_{di}^{k-pj-q+1}(\beta)|V_i[pj+q-1]).$$

Consequently, for $j \in \{1, 2, 3, ...\}$ the following equalities hold:

$$\begin{aligned}
V_i[pj] &= F_i[pj]F_i[p(j-1)+q-1]...F_i[p(j-1)+1]|A_{di}^p(\beta)| \\
&\quad \times ...F_i[2p]F_i[p+q-1]...F_i[p+1]|A_{di}^p(\beta)|F_i[p]F_i[q-1]...F_i[1]|A_{di}^p(\beta)|V_i[0] \\
V_i[pj] &= F_i[pj]|A_{di}(\beta)|...F_i[p(j-1)+1]|A_{di}(\beta)||A_{di}^{p-q}(\beta)| \\
&\quad \times F_i[2p]|A_{di}(\beta)|...F_i[p+1]|A_{di}(\beta)||A_{di}^{p-q}(\beta)|F_i[p]|A_{di}(\beta)|...F_i[1]|A_{di}(\beta)||A_{di}^{p-q}(\beta)|V_i[0] \\
&\Rightarrow \\
E[V_i^2[pj]] &= (E[F_i^2[1]|A_{di}(\beta)|^2])^{qj}|A_{di}^{j(p-q)}(\beta)|E[V_i^2[0]] \\
E[V_i^2[pj]] &= ((1-\alpha_i)\frac{|A_{di}(\beta)|^2}{2^{2\mathcal{R}_i}} + \alpha_i|A_{di}(\beta)|^2)^{qj}|A_{di}^{j(p-q)}(\beta)|E[V_i^2[0]] \\
E[V_i^2[pj]] &= (|A_{di}(\beta)|^{p+q}((1-\alpha_i)\frac{1}{2^{2\mathcal{R}_i}} + \alpha_i)^q)^j|E[V_i^2[0]].
\end{aligned}$$

Now, as it is assumed for each i that we have $|A_{di}(\beta)|^{p+q}((1-\alpha_i)\frac{1}{2^{2\mathcal{R}_i}} + \alpha_i)^q < 1$, $E[V_i^2[pj]]$ along with the sequence $E[V_i^2[pj+1]]$, ... , $E[V_i^2[pj+p-1]]$ converge to zero as $j \to \infty$. This completes the proof as for each i we have $E[|x_i[k] - \hat{x}_i[k|k]|^2] \leq E[V_i^2[k]]$, in which under the assumption of proposition, we have $E[V_i^2[k]] \to 0$ and hence $E[|x_i[k] - \hat{x}_i[k|k]|^2] \to 0$.

Again, let $e[k] = x[k] - \hat{x}[k]$ denote the estimation error where $\hat{x}[k]$ is the output of the one step ahead estimator block of Fig. 2, which is updated as follows:

$$\hat{x}[k+1] = A_d(\beta)\hat{x}[k|k] + B_d(\beta)u[k].$$

Then, we have the following corollary.

Corollary 3.6 *i) From the definition of $e[k]$ we have the following inequality for $E||e[k]||^2$*

$$E||e[k]||^2 = \sigma_{max}^2(A_d(\beta))\sum_{i=1}^{n} E||x_i[k] - \hat{x}_i[k|k]||^2.$$

Hence, from the above proposition it follows that if the condition (12) holds, then $E||e[k]||^2$ converges to zero, independent of the control action and the state variables, with an exponential rate. Hence, as mean square convergence implies mean absolute convergence [18], under the assumption of Proposition 3.5, $E||e(k)|| \to 0$, as $k \to \infty$.
ii) The above coding strategy has a semi recursive structure; and hence, it has very low coding computational complexity.

3.3 Coding Strategy - $\beta = 1$

For $p = q = 1$, i.e., $\beta = 1$, the coding strategy 3.2 is reduced to the coding strategy that uses feedback channel full time and result in asymptotic mean absolute tracking of the state trajectory. For this case, the mean absolute estimation error converges to zero, as time increases if the following inequality holds for all $i \in \{1, 2, ..., n\}$ and $h \in \{1, 2, ..., n_i\}$

$$(1 - \alpha_i)\frac{|\lambda_h(A_{di}(\beta))|^2}{2^{2\mathcal{R}_{ih}}} + \alpha_i|\lambda_h(A_{di}(\beta))|^2 < 1.$$

4 Stability and Performance Results

In this section, it is shown that for a given β, under some conditions, asymptotic mean absolute stability is achieved if the corresponding coding strategy of Section 3 is combined with the controller (3). This result is shown in the following theorem.

Theorem 4.1 *Consider the closed loop feedback system of Fig. 2, which is described by either the coding strategy 3.1 (i.e., $\beta = 0$), coding strategy 3.2 with $\beta \in (0, 1)$, or the coding strategy 3.3 (i.e., $\beta = 1$) and the controller (3), where the controller gain K_β is chosen such that the matrix $A_d(\beta) + B_d(\beta)K_\beta$ is a stable matrix. Suppose that the coding strategy results in asymptotic mean absolute tracking of state trajectory at controller and computation overflow does not occur (i.e., the conditions (4) and (5) are satisfied). Then, the system is asymptotically mean absolute stable.*

Proof: Consider the feedback system of Fig. 2 equipped with one of the above coding strategies. Recall that the state estimate to be used in the stabilizing controller is given by

$$\hat{x}[k+1] = A_d(\beta)\hat{x}[k|k] + B_d(\beta)u[k],$$

where the vector $\hat{x}[k|k] = \begin{pmatrix} \hat{x}_1[k|k] \\ . \\ . \\ . \\ \hat{x}_n[k|k] \end{pmatrix}$ is given either by (7) or (11) depending on the duty cycle for feedback channel that is used. Recall also that $e[k] = x[k] - \hat{x}[k]$ is the estimation error.

Now, for the corresponding closed loop feedback system, the following equalities hold

$$x[k+1] = A_d(\beta)x[k] + B_d(\beta)u[k],$$

$$u[0] = 0,\; u[k+1] = K_\beta \hat{x}[k+1], \quad \hat{x}[k+1] = A_d(\beta)\hat{x}[k|k] + B_d(\beta)u[k], k \in \mathbf{N}_+.$$

Subsequently, for $k \geq 1$, we have:

$$x[1] = A_d(\beta)x[0],\; x[k+1] = (A_d(\beta) + B_d(\beta)K_\beta)x[k] - B_d(\beta)K_\beta e[k].$$

Therefore, for $k \geq 2$, the following equality holds

$$x[k] \;\; = \;\; (A_d(\beta) + B_d(\beta)K_\beta)^{k-1}A_d(\beta)x[0] - \sum_{j=1}^{k-1}(A_d(\beta) + B_d(\beta)K_\beta)^{k-1-j}B_d(\beta)K_\beta e[j].$$

Consequently, for $k \geq 2$, the following inequality holds

$$\begin{aligned} E||x[k]|| \;\; \leq \;\; & (\sigma_{max}(A_d(\beta) + B_d(\beta)K_\beta))^{k-1}\sigma_{max}(A_d(\beta))E||x[0]|| \\ & + \sum_{j=1}^{k-1}(\sigma_{max}(A_d(\beta) + B_d(\beta)K_\beta))^{k-1-j}\sigma_{max}(B_d(\beta)K_\beta)E||e[j]||. \end{aligned} \tag{13}$$

Now, as the matrix $A_d(\beta) + B_d(\beta)K_\beta$ is stable and, as shown in Section 3.1, $E||e[j]||$ converges to zero, the first and the second terms of the right hand side of (13) vanish as time increases. That is, $\lim_{k\to\infty} E||x[k]|| = 0$.

As pointed out in Remark 3.3, among the available coding strategies that do not use feedback channel and provide reliable tracking for dynamic systems via estimating the initial state, the coding strategy presented in Section 3.1 for the case of $\beta = 0$ has the fastest decoding error decay rate. Moreover, the coding strategies, as proposed for the other cases, i.e., $\beta = 1$ and $\beta = \frac{q}{p}$ ($p, q \in \{1, 2, 3, ...\}$, $q < p$) have a fast decoding error decay rate as they have exponential decay rates. In addition, as shown in this section, for each β there may exist many controller gains K_β that guarantee asymptotic mean absolute stability. Now, the question is which gain results in the best performance? To address this question, we define the parameter $T^\epsilon(\beta)$, which is the smallest time instant under which for the given $\epsilon > 0$, we have $E||x[k]|| \leq \epsilon$, $\forall k \geq T^\epsilon(\beta)$.

As will be shown in the following, for a given β, in order to have the best performance using the proposed stabilizing technique, the stabilizing controller (3) with gain K_β that results in the smallest possible $\sigma_{max}(A_d(\beta) + B_d(\beta)K_\beta)$ must be used.

Theorem 4.2 *Consider the closed loop feedback system of Fig. 2, in which for each β it is described by the corresponding pairs of encoders-decoders $(\mathcal{E}_{i\beta}(\cdot), \mathcal{D}_{i\beta}(\cdot))$ of Section 3, and the controller (3). Then, an upper bound, $T_{up}^{\epsilon}(\beta)$ on $T^{\epsilon}(\beta)$ is the smallest integer $T_{up}^{\epsilon}(\beta)$ that satisfies the following inequality for all $k \geq T_{up}^{\epsilon}(\beta)$.*

$$\begin{aligned}&\Big(\sigma_{max}(A_d(\beta)+B_d(\beta)K_\beta)\Big)^{k-1}\sigma_{max}(A_d(\beta))L[0]\\&+\sum_{j=1}^{k-1}\Big(\sigma_{max}(A_d(\beta)+B_d(\beta)K_\beta)\Big)^{k-1-j}\sigma_{max}(B_d(\beta)K_\beta)E||e(j)|| \leq \epsilon, \qquad (14)\end{aligned}$$

where $e[j] \doteq x[j]-\hat{x}[j]$ is the estimation error associated with the one step ahead estimator block of Fig. 2 and $L[0] \doteq \sqrt{\sum_{i=1}^{n}(L_i[0])^2}$, in which the non-negative scalars $L_i[0]$, $i \in \{1,2,...,n\}$, are known upper bounds on the initial state (i.e., $||x_i[0]|| \leq L_i[0]$).

Proof: Recall that $x[k+1] = A_d(\beta)x[k] + B_d(\beta)u[k]$. Therefore, the following equality holds

$$x[k+1] = (A_d(\beta)+B_d(\beta)K_\beta)x[k] - B_d(\beta)K_\beta e[k], \quad k \geq 1.$$

Consequently, for $k \geq 1$ the following relationships hold

$$\begin{aligned}x[k] &= \Big(A_d(\beta)+B_d(\beta)K_\beta\Big)^{k-1}A_d(\beta)x[0] - \sum_{j=1}^{k-1}\Big(A_d(\beta)+B_d(\beta)K_\beta\Big)^{k-1-j}B_d(\beta)K_\beta e[j]\\||x[k]|| &\leq \Big(\sigma_{max}(A_d(\beta)+B_d(\beta)K_\beta)\Big)^{k-1}\sigma_{max}(A_d(\beta))||x[0]||\\&\quad+\sum_{j=1}^{k-1}\Big(\sigma_{max}(A_d(\beta)+B_d(\beta)K_\beta)\Big)^{k-j-1}\sigma_{max}(B_d(\beta)K_\beta)||e[j]||.\end{aligned}$$

Now, from the inequality (14) it follows that $E||x[k]|| \leq \epsilon$, $\forall k \geq T_{up}^{\epsilon}(\beta)$. That is, an upper bound on $T^{\epsilon}(\beta)$ is $T_{up}^{\epsilon}(\beta)$ that satisfies (14).

Remark 4.3 *i) When subsystems are scalar, for the case of $\beta \neq 0$, from Corollary 3.6, i, it follows that $||e[j]|| \leq \sigma_{max}(A_d(\beta))\sqrt{\sum_{i=1}^{n}V_i[j]}$ (where an expression for $V_i[j]$ was given in the proof of Proposition 3.5). Hence, $E||e[j]||$ in (14) can be replaced by $\sigma_{max}(A_d(\beta))\sqrt{\sum_{i=1}^{n}V_i[j]}$. That is, for $\beta = 1$ and $\beta = \frac{q}{p}$ ($p,q \in \{1,2,3,...\}$, $q<p$), the upper bound $T_{up}^{\epsilon}(\beta)$ is the smallest integer that satisfies the following inequality for all $k \geq T_{up}^{\epsilon}(\beta)$*

$$\begin{aligned}&\Big(\sigma_{max}(A_d(\beta)+B_d(\beta)K_\beta)\Big)^{k-1}\sigma_{max}(A_d(\beta))L[0]\\&+\sum_{j=1}^{k-1}\Big(\sigma_{max}(A_d(\beta)+B_d(\beta)K_\beta)\Big)^{k-j-1}\sigma_{max}(B_d(\beta)K_\beta)\sigma_{max}(A_d(\beta))\sqrt{\sum_{i=1}^{n}V_i[j]} \leq \epsilon (15)\end{aligned}$$

ii) For the case of $\beta = 0$ from the inequality (10) it follows that the condition (14) can be replaced by

$$\Big(\sigma_{max}(A_d(\beta) + B_d(\beta)K_\beta)\Big)^{k-1}\sigma_{max}(A_d(\beta))L[0]$$
$$+\sum_{j=1}^{k-1}\Big(\sigma_{max}(A_d(\beta) + B_d(\beta)K_\beta)\Big)^{k-j-1}\sigma_{max}(B_d(\beta)K_\beta)$$
$$\times\sigma_{max}(A_d(\beta))\sqrt{\sum_{i=1}^{n} c_i^2\frac{j-1}{2^{(\Delta(R_i,n_i,\alpha_i)-2\log\sigma_{max}(A_{di}(\beta)))(j-1)}}} \le \epsilon. \qquad (16)$$

iii) Note that the settling time $T_s^\epsilon(\beta)$ can be approximated as follows: $T_s^\epsilon(\beta) \approx T_\beta T^\epsilon(\beta)$. Hence, a scaled version of $T_{up}^\epsilon(\beta)$ (i.e., $T_\beta T_{up}^\epsilon(\beta)$) represents an upper bound on the settling time.

Following the above remarks, for a given β by choosing the controller gain K_β that results in the smallest possible $\sigma_{max}(A_d(\beta)+B_d(\beta)K_\beta)$, and using the corresponding coding strategy, as given in Section 3, we can have the smallest possible upper bound $T_{up}^\epsilon(\beta)$, as given by (15) for the cases of $\beta = 1$ and $\beta = \frac{q}{p}$ ($p, q \in \{1, 2, 3, ...\}$, $q < p$) and (16) for the case of $\beta = 0$, using the proposed coding strategies and controller.

Following the above results, we have the following corollary, which summarizes the trade-offs between duty cycle for feedback channel use, transmission delay and performance.

Corollary 4.4 *i) As discussed in Remark 3.3 for $\beta = 0$, $e[j]$ decays to zero very slowly such that for $\sigma_{max}(A_d(\beta))$ with unstable $A_d(\beta)$, the coding strategy of Section 3.1 may not even guarantee reliable tracking of state trajectory at controller. For this case (i.e., $\beta = 0$), the sampling period T_β is large due to high coding computational complexity. This results in a large $\sigma_{max}(A_d(\beta))$ for an unstable $A_d(\beta)$. Therefore, from (16), it follows that if this coding strategy guarantees reliable tracking, then the upper bound on the settling time can be very large for a given ϵ.*
ii) For $\beta = 1$ and $\beta = \frac{q}{p}$ ($p, q \in \{1, 2, 3, ...\}$, $q < p$), $\sum_{i=1}^{n} V_i[j]$ will decay to zero quickly. Also, due to relatively low coding computational complexity of the corresponding coding strategy, the sampling period T_β is relatively small. This results in a relatively small $\sigma_{max}(A_d(\beta))$ for an unstable $A_d(\beta)$. Therefore, from (15) it follows that the upper bound on the settling time for $\beta = 1$ or $\beta = \frac{q}{p}$ ($p, q \in \{1, 2, 3, ...\}$, $q < p$) is relatively smaller than that of the case of $\beta = 0$.

For the purpose of illustration in the following we consider the closed loop feedback system of Fig. 2 described by the following two subsystems.

$$\begin{cases} \dot{x}_1(t) = 3x_1(t) + 3u(t), \\ y_1(t) = x_1(t),\ x_1(0) \sim U(0, 1), \end{cases}$$

$\beta = 0$	$T_s^\epsilon(\beta) = \infty$
$\beta = 1/4$	$T_s^\epsilon(\beta) = 10.17$ sec.
$\beta = 1/3$	$T_s^\epsilon(\beta) = 9.28$ sec.
$\beta = 1/2$	$T_s^\epsilon(\beta) = 6.62$ sec.
$\beta = 2/3$	$T_s^\epsilon(\beta) = 6.52$ sec.
$\beta = 1$	$T_s^\epsilon(\beta) = 6.52$ sec.

Table 1: Trade-off between $\beta \in \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$ and the settling time.

$$\begin{cases} \dot{x}_2(t) = 3.6x_2(t) - 3u(t), \\ y_2(t) = x_2(t), \;\; x_2(0) \sim U(0,1), \end{cases} \tag{17}$$

Here, it is assumed that $\epsilon = 0.02$, $\alpha_1 = 0.1$, $\alpha_2 = 0.1$, and $K_\beta = [a_1 \; a_2]$, $-10 \leq a_1, a_2 \leq 10$. It is also assumed that the transmission delay is $T_{di} = 0.04$ second, $i \in \{1,2\}$. Moreover, it is assumed that the communication bit rate is large enough such that there is no transmission delay due to the terms $\frac{\mathcal{R}_{ik}}{BW}$ and $\frac{\mathcal{R}_u+n}{BW}$. For the stability and tracking of the dynamic system (17) we apply the developed coding strategies and controller with different duty cycle for feedback channel use β. For $\beta = 0$, we use the coding strategy of Section 3.1 with $R_1 = R_2 = 1$. For this case it is observed that $\mathcal{C}_d(\beta) = 0.03$s and $\mathcal{C}_c(\beta) = 0.02$s. Therefore, following the conditions (4) and (5), we set $T_m = 0.02$s; and hence, $T_\beta = 0.12$s. However, for this time period it is observed that the coding strategy is not able to provide reliable tracking; and therefore, we are not able to stabilize the controlled system using the proposed coding strategy and controller. This result is expected from Corollary 3.4 as for $n_i = 1$, $\alpha_i = 0.1$, $R_i = 1$, we have $\Delta(R_i, n_i, \alpha_i) = 0$; and hence, the condition $\Delta(R_i, n_i, \alpha_i) > 2\log\sigma_{max}(A_{di}(\beta))$ does not hold. In fact, for this system the maximum value of $\Delta(R_i, n_i, \alpha_i)$ is achieved for $R_i = 0.28$. For $R_i = 0.28$, we have $\Delta(R_i, n_i, \alpha_i) = 0.28$. Nevertheless, considering only transmission delay, the smallest value for $\sigma_{max}(A_{di}(\beta))$ is equivalent to $e^{3.6\times0.08} = 1.3312$, which indicates that the condition $\Delta(R_i, n_i, \alpha_i) > 2\log\sigma_{max}(A_{di}(\beta))$ does not also hold for other R_is.

For $\beta > 0$, we use the coding strategy of Section 3.2 with information rates $\mathcal{R}_1 = \mathcal{R}_2 = 6$ bits. For the cases of $\beta \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$ it is observed that the coding computational complexity is negligible (it is of the order of 10^{-4}s); and hence, we chose $T_\beta = 0.08$ second. For these cases, K_β that results in the smallest $\sigma_{max}(A_d(\beta) + B_d(\beta)K_\beta)$ is calculated as follows: $K_\beta = (\, 6.9 \quad 9.4 \,)$.

In Table 1, we summarized trade-off between $\beta \in \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$ and the settling time. In Fig. 4, we also illustrated this trade-off. From Fig. 4 it is clear that for the condition simulated, almost noting is lost in terms of performance if $\beta \in [\frac{1}{2}, 1]$.

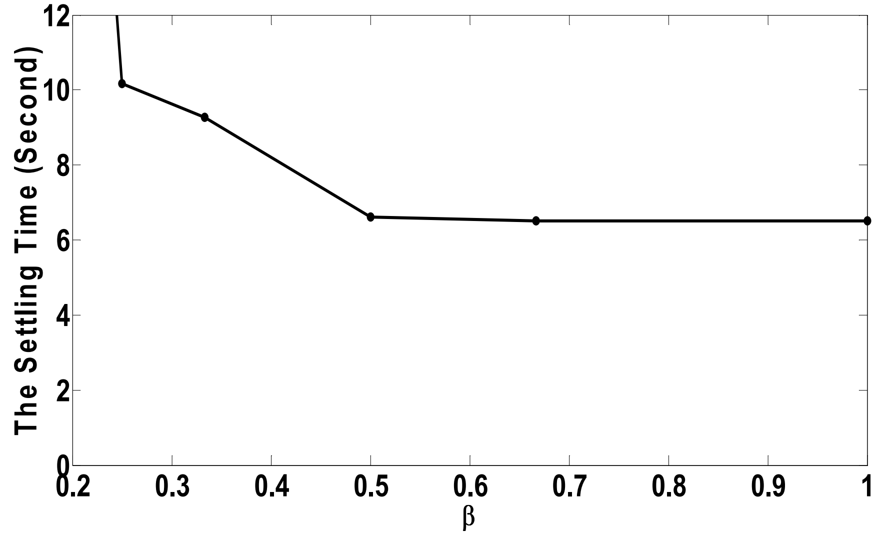

Figure 4: Trade-off between β and the settling time.

5 Conclusion

Many results in the literature (e.g., [11],[12],[13],[14],[15],[16],[17]) have been developed under the assumption of full time availability of noiseless feedback channel. Specifically, there is not any result for stability and tracking with intermittent noiseless feedback channel over the packet erasure channel, which is an important class of digital communication channels as it is an abstract model for the commonly used information technologies, such as the Internet, WiFi and mobile communication. However, the availability of noiseless feedback channel may not be possible full time. The full time availability of noiseless feedback channel requires that feedback channel signal is transmitted with high power full time. This results in significant power consumption at the base station of the control system of Fig. 1 as the control signal in this system is also transmitted with high power full time. Hence, as transmission of two high power signals full time will result in a short life time for the transmitter of the high level controller, in this paper we relaxed the full time availability assumption of noiseless feedback channel and addressed the problem of tracking state trajectory at remote controller, stability and performance of linear time-invariant noiseless dynamic systems with multiple observations over the packet erasure network subject to transmission delay that does not necessarily use feedback channel full time. Three cases were considered in this paper: i) without feedback channel, ii) with feedback channel intermittently, and iii) with full time availability of feedback channel. For all three cases, coding strategies that result in asymptotic mean absolute tracking were presented. Asymptotic mean absolute stability of the controlled system equipped with each of these coding strategies was also shown. Moreover, trade-offs between duty cycle for feedback channel use, transmission delay and performance were studied.

Dynamic systems can be viewed as continuous alphabet sources with memory. Consequently, many works in the literature (e.g.,[1],[15],[16],[17]) are concerned with the question of stability and tracking over AWGN channel which itself is naturally a continuous alphabet channel. In [15],[16] and [17] the authors addressed the problem of mean square stability and tracking of linear unstable Gaussian systems (i.e., systems with terms that are not known for design) under the assumption of full time availability of noiseless feedback channel. For future, it is interesting to relax the assumption of full time availability of noiseless feedback channel and address the stability and tracking of linear Gaussian systems over AWGN channel when feedback channel is available intermittently. Another research direction is to address the stability and tracking problem of nonlinear Lipschitz systems with input affine structure when feedback channel is available intermittently. As shown in [11], due to the globally Lipschitz property of such systems, the developed techniques for stability and tracking of linear systems can be easily extended to this class of nonlinear systems. Hence, another research direction is to extend the developed techniques in this paper to nonlinear noiseless Lipschitz systems with input affine structure over the packet erasure network with intermittent noiseless feedback channel. Other research direction is to extend the developed techniques in this paper to nonlinear noiseless smooth systems via linearizing the nonlinear system around the working points. These problems are left for future investigation.

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6 Appendix

In this section, we recall the coding strategy of [10] that is used to reconstruct a bounded initial state, reliably, over the binary erasure channel.

Without loss of generality and for the simplicity of presentation, consider the following unknown initial state

$$x[0] = \begin{pmatrix} x_1[0] \\ \vdots \\ x_m[0] \end{pmatrix}$$

with $x_h[0] \in [0,1]$, $h \in \{1,2,...,m\}$. For a system with $x_h[0] \notin [0,1]$, we use the following transformation: $r[0] = \Phi(x[0] - E)$, where the matrix Φ is invertible and Φ and E are chosen such that $r_h[0] \in [0,1]$.
$x_h[0] \in [0,1]$ has the following binary representation:

$$x_h[0] = \sum_{j=1}^{\infty} w_{hj} 2^{-j}, \quad w_{hj} \in \{0,1\}.$$

At time instant k, for each h, the codeword/packet $\delta_h[k]$ with length $\mathcal{R}_k = \lfloor R.(k+1) \rfloor$, $0 < R \leq 1$ are produced from the following linear operation:

$$\begin{pmatrix} \delta_h[0] \\ \delta_h[1] \\ \vdots \\ \delta_h[k] \\ \vdots \end{pmatrix} = M \bigoplus \begin{pmatrix} w_{h1} \\ w_{h2} \\ \vdots \\ w_{hl_h} \\ \vdots \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & & \end{pmatrix},$$

where 0s and 1s on the lower triangular side of the matrix M are generated randomly; but transmitter and receiver know the components of this matrix a priori. In the above equation, the operator $\bigoplus$ acts as follows:

$$\delta_h[0] = w_{h1}, \ \delta_h[1] = (0 \ w_{h2}), \ ..., \ \delta_h[k] = (w_{h1}...w_{hl_h}), ... \ .$$

Then, for each h, the corresponding packet $\delta_h[k]$ is transmitted bit by bit via the binary erasure channel at time instant k. At time instant k, the decoder can use all the received packets up to time k, i.e., $\bar{\delta}_h[0]$, $\bar{\delta}_h[1]$, ..., $\bar{\delta}_h[k]$ to reconstruct $\hat{x}_h[0|k]$, which is the estimate of the hth component of the initial state $x[0]$ at time instant k. To achieve this goal, the decoder ignores the packets containing filliped bits that cannot be recovered and produces $\hat{x}_h[0|k]$ using only the packets that received successfully. To understand how the decoding operation works, let us assume that for the hth component, just the received codeword $\bar{\delta}_h[1]$ contains erased bits. Then, the decoder ignores $\bar{\delta}_h[1]$ and uses the following linear system of equations to reconstruct $\hat{x}_h[0|k]$.

$$\begin{pmatrix} z_h[0] \\ z_h[2] \\ \vdots \\ z_h[k] \end{pmatrix} = \bar{M} \begin{pmatrix} w_{h1} \\ w_{h2} \\ \vdots \\ w_{hl_h} \end{pmatrix},$$

where $z_h[k]$ is the decimal representation of the binary string $\delta_h[k]$, and $\bar{M}$ is the matrix M without the second row (which corresponds to $\delta_h[1]$) with $\frac{1}{2^p}$ corresponding to each 1 located at the pth column of the matrix M. For illustration, for M given above,

$$\bar{M} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ \frac{1}{2} & \dots & \frac{1}{2^p} & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & & \end{pmatrix}.$$

Consequently, using the above equation, the decoder estimates w_{h1}, w_{h2}, ..., w_{hl_h} as follows:

$$\begin{pmatrix} \hat{w}_{h1} \\ \hat{w}_{h2} \\ \vdots \\ \hat{w}_{hl_h} \end{pmatrix} = (\bar{M}'\bar{M})^{-1}\bar{M}' \begin{pmatrix} z_h[0] \\ z_h[2] \\ \vdots \\ z_h[k] \end{pmatrix}.$$

Subsequently, it outputs: $\hat{x}_h[0|k] = \sum_{j=1}^{l_h} \hat{w}_{hj} 2^{-j}$.

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