Robust Entropy Rate for Uncertain Sources: Applications to Communication and Control Systems

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***Abstract*— In this paper the notion of robust entropy and subsequently, robust entropy rate for a family of discrete time uncertain sources is introduced. When the uncertainty is described by a relative entropy constraint between the set of uncertain source densities and a given nominal source density, the solution to this robust notion of information is presented and its connection with other notions of entropy definitions, such as, Renyi entropy and Tsallis entropy is presented. Then, the robust entropy rate is calculated for 1) Uncertain sources corresponding to a partially observed Gauss Markov process, 2) Sources with uncertain frequency response, and 3) Uncertain sources corresponding to a partially observed controlled Gauss Markov Process. Finally, an application of the robust entropy rate in networked control systems is presented by defining necessary conditions for uniform asymptotic stabilizability and observability.**

I. Introduction

The entropy and entropy rate are information theoretic measures. They have applications in physics, probability and statistics, communication theory and economics. The importance of entropy in communication theory was first introduced by Shannon in terms of Shannon first coding theorem. Then, the application of entropy rate in joint source channel coding theorem, the AEP and etc. is shown [1].

Let $f(y)$ represent the Probability Density Function (PDF) corresponding to a random variable $Y \in \Re^d$. The Shannon entropy is defined by $H_S(f) \stackrel{\triangle}{=} -\int f(y)\ln f(y)dy$ ($f \in L_1$). In addition to the Shannon entropy, there are the Renyi, defined by $H_R \stackrel{\triangle}{=} \frac{1}{1-\alpha}\ln\int f^{\alpha}(y)dy$, for $\alpha > 0$, and $\alpha \neq 1$ ($f^{\alpha} \in L_1$) [2], and Tsallis entropy [3] defined by $H_T(f) \stackrel{\triangle}{=} \frac{1}{1-\alpha}(\int f^{\alpha}(y)dy - 1)$. Tsallis entropy gives us as special case the Shannon entropy, in particular, as $H_S(f) = \lim_{\alpha\to 1} H_R(f)$, and since $H_T(f) = \frac{1}{1-\alpha}\{e^{(1-\alpha)H_R(f)} - 1\}$, by expanding the exponential term, $e^{(1-\alpha)H_R(f)}$, and taking the limit as $\alpha \to 1$, we get $H_S(f) = \lim_{\alpha\to 1} H_T(f) = \lim_{\alpha\to 1} H_R(f)$.

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The objective of this paper is to extend the notion of entropy and subsequently entropy rate to the case when there is uncertainty in the source. The robust entropy is defined as the maximum of the Shannon entropy over a family of sources belonging to an uncertainty set. The explicit solution to the robust entropy is presented when the uncertainty is described by a constraint on the relative entropy between the set of uncertain source densities and the corresponding nominal source density. Subsequently, the connection between this solution with other entropies is shown. Then, for different families of uncertain source densities, the robust entropy rate is calculated and an application of the robust entropy rate in stabilizability and observability of networked control systems is presented.

This paper is organized as follows. In Section II, the robust entropy and the robust entropy rate are defined. The solution to the robust entropy and its connection to other kinds of entropy are presented. In Section III, for different families of uncertain sources, the robust entropy rate is calculated. Finally in Section IV, an application of robust entropy rate in stabilizability and observability of networked control system is presented.

II. Problem formulation, solution and connections

Let $\mathcal{D}$ denote the space of density functions defined on $\Re^d$. In real world situation, the source is not entirely known. This introduces some degree of uncertainty in source density around a nominal fixed source density $g(y)$. Let the true source density, $f(y)$, belongs to the uncertainty set $\mathcal{D}_{SU} \subset \mathcal{D}$. Then we have the following definition for robust entropy and subsequently for robust entropy rate.

Definition 2.1: Let Y be a random variable (or a sequence of R.V's.) and $f(y)$ the corresponding density associated with the uncertain source such that $f(y) \in \mathcal{D}_{SU}$. Then the robust entropy of Y is defined by

$$H_{robust}(f^*) = \sup_{f\in\mathcal{D}_{SU}} H_S(f), \tag{1}$$

where $f^* = argsup_{f\in\mathcal{D}_{SU}} H_S(f)$.

Moreover, if $Y = (Y_0, ..., Y_{T-1})^{tr}$ represents a sequence of R.V's. with length T of source symbols produced by the uncertain source with joint density $f \in \mathcal{D}_{SU}$, the robust entropy rate is defined by

$$\mathcal{H}_{robust}(\mathcal{Y}) = \lim_{T\to\infty}\frac{1}{T}H_{robust}(f^*), \tag{2}$$

provided the limit exists.

Remark 2.2: For the case without uncertainty ($\mathcal{D}_{SU} = \{g\}$), the robust entropy and robust entropy rate are reduced to the Shannon entropy and entropy rate.

A. Relative Entropy Uncertainty Set

Throughout this section we consider an uncertain set defined by

$$\mathcal{D}_{SU} \triangleq \{f \in \mathcal{D}; H(f|g) \leq R_c\}, \quad (3)$$

where $H(.|.)$ is the relative entropy and $R_c \in [0, \infty)$ is fixed.

Lemma 2.3: [4] Given a fixed nominal density $g(y)$ and the uncertainty set $\mathcal{D}_{SU}$, the robust entropy is given by

$$\begin{aligned} &H_{robust}(f^{*,s^*}) = \\ &min_{s\geq 0}[sR_c + (1+s)\ln\int g(y)^{\frac{s}{1+s}}dy] \\ &\triangleq H_{robust}^{R_c}(f^{*,s^*}) \end{aligned} \quad (4)$$

and

$$f^{*,s}(y) = \frac{g(y)^{\frac{s}{1+s}}}{\int g(y)^{\frac{s}{1+s}}dy}, \quad (5)$$

where the minimizing $s^* \geq 0$ in (4) and (5) is the unique solution of $H(f^{*,s^*}|g) = R_c$.

Remark 2.4: The above solution for the robust entropy is related to the Renyi and consequently to the Tsallis entropies as follows. Let $\alpha = \frac{s}{1+s}$; then

$$H_{robust}^{R_c}(f^{*,s^*}) = min_{\alpha\in[0,1)}\Big\{\frac{\alpha}{1-\alpha}R_c + H_R(g)\Big\}. \quad (6)$$

Moreover, it can be shown that

$$\begin{aligned} min_{\alpha\in[0,1)}H_R(g) &\leq H_{robust}^{R_c}(f^{*,s^*}) \\ &\leq \frac{\alpha}{1-\alpha}R_c + H_R(g), \ \alpha \in [0,1). \end{aligned} \quad (7)$$

Corollary 2.5: [5] Suppose $R_c \leq H(h|g)$, where $h(y)$ is a uniform Probability Mass Function (PMF) (e.g., $h(y) = \sum_{i=1}^M h(y_i)\delta(y_i), h(y_i) = \frac{1}{M}$ and $\delta(.)$ is a delta measure). When $g(y)$ and consequently $f(y)$ correspond to PMF's, that is, $g(y) = \sum_{i=1}^M g(y_i)\delta(y_i)$ and $f(y) = \sum_{i=1}^M f(y_i)\delta(y_i)$, then (4) and (5) are reduced to

$$\begin{aligned} H_{robust}^{R_c}(f^{*,s^*}) &= min_{s\geq 0}[sR_c + (1+s) \\ &. \ln\sum_{i=1}^M g(y_i)^{\frac{s}{1+s}}], \end{aligned} \quad (8)$$

$$f^{*,s}(y_i) = \frac{g(y_i)^{\frac{s}{1+s}}}{\sum_i g(y_i)^{\frac{s}{1+s}}}, \quad 1 \leq i \leq M, \quad (9)$$

where the minimizing $s^* \geq 0$ in (8) and (9) is the unique solution of $H(f^{*,s^*}|g) = R_c$.

Next, the robust entropy rate is computed as a direct consequence of Lemma 2.3.

Corollary 2.6: Let $Y = (Y_0, ..., Y_{T-1})^{tr}$ be a sequence with length T of source symbols with uncertain joint density function $f(y) \in \mathcal{D}_{SU}$, $y \in \Re^{Td}$ and $R_c \rightarrow TR_c$.

The robust entropy rate is given by

$$\begin{aligned} \mathcal{H}_{robust}(\mathcal{Y}) &= \lim_{T\rightarrow\infty}\frac{1}{T}H_{robust}^{TR_c}(f^{*,s^*}), \\ H_{robust}^{TR_c}(f^{*,s^*}) &= min_{s\geq 0}[sTR_c + (1+s) \\ &. \ln\int g(y)^{\frac{s}{1+s}}dy] \triangleq H_{robust}^{TR_c}(f^{*,s^*}), \end{aligned} \quad (10)$$

and

$$f^{*,s}(y) = \frac{g(y)^{\frac{s}{1+s}}}{\int g(y)^{\frac{s}{1+s}}dy}, \quad (11)$$

where the minimizing $s^* \geq 0$ in (10) and (11) is the unique solution of $H(f^{*,s^*}|g) = TR_c$.

Remark 2.7: Clearly, $\lim_{T\rightarrow\infty}\frac{1}{T}H_{robust}^{TR_c}(f^{*,s^*})$, is the solution of the following robust entropy rate

$$\lim_{T\rightarrow\infty}\sup_{\{f\in\mathcal{D};\frac{1}{T}H(f|g)\leq R_c\}}\frac{1}{T}H_S(f). \quad (12)$$

Example 2.8: [4] From Corollary 2.6 it follows that, if the nominal source density $g(y)$ is Td-dimensional Gaussian density function with mean m and covariance Γ_Y, $\forall R_c \in [0, \infty)$,

$$\begin{aligned} \frac{1}{T}H_{robust}^{TR_c}(f^{*,s^*}) &= \frac{d}{2}\ln(\frac{1+s}{s}) + \frac{d}{2}\ln(2\pi e) \\ &+ \frac{1}{2T}\ln\det\Gamma_Y, \end{aligned} \quad (13)$$

where $s > 0$ is the unique solution of the following nonlinear equation

$$R_c = -\frac{d}{2}\ln(\frac{1+s}{s}) + \frac{d}{2s}. \quad (14)$$

III. Robust Entropy Rate Calculation

In this section, the robust entropy rate is calculated for 1) Uncertain sources corresponding to a partially observed Gauss Markov process, 2) Sources with uncertain frequency response, and 3) Uncertain sources corresponding to a partially observed controlled Gauss Markov process.

A. Partially Observed Gauss Markov Process

In this section, the uncertainty is described by a constraint on the relative entropy between the set of uncertain sources and the corresponding nominal source via

$$\mathcal{D}_{SU} = \{f \in \mathcal{D}; \frac{1}{T}H(f|g) \leq R_c\}, \quad (15)$$

where $f(y)$ and $g(y)$ are PDF's corresponding to a sequence with length T of the symbols produced by uncertain and nominal sources, respectively.

Now, consider a nominal density induced by a partially observed Gauss Markov nominal source described via

$$\begin{aligned} X_{t+1} &= AX_t + BW_t, \quad X_0, \\ Y_t &= CX_t + DV_t, \ t \in \mathcal{N}_+ \triangleq \{0,1,2,...\}, \end{aligned} \quad (16)$$

where $X_t \in \Re^n$ denotes the unobserved process, $Y_t \in \Re^d$ is the observed process, $W_t \in \Re^m$, $V_t \in \Re^l$, W_t is i.i.d. $\sim N(0, I_{m\times m})$, V_t is i.i.d. $\sim N(0, I_{l\times l})$, $X_0 \sim N(\bar{x}_0, V_0)$, and $\{X_0, V_t, W_t\}$ are mutually independent, $t \in \mathcal{N}_+$. Here it is assumed that (C, A) is detectable, $(A, (BB^{tr})^{\frac{1}{2}})$ is stabilizable and $D \neq 0$.
The objective is to calculate the robust entropy rate. We shall need the following lemmas.

Lemma 3.1: [4] Let $Y : \Omega \times \mathcal{N}_+ \to \Re^d$, be a stationary Gaussian process with power spectral density $S_Y(e^{jw})$. Let $Z_t \triangleq Y_t - E[Y_t|Y^{t-1}]$, $Y^{t-1} = \{Y_0, ..., Y_{t-1}\}$, $\Lambda_t \triangleq Cov(Z_t)$ and assume $\Lambda_\infty \triangleq \lim_{t\to\infty} \Lambda_t$ exists. Then an application of Szego limit formula [6] and Cholsky decomposition [7] implies that

$$\begin{aligned}\ln\det\Lambda_\infty &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S_Y(e^{jw})dw \\ &= \lim_{T\to\infty}\frac{1}{T}\ln\det\Gamma_Y, \end{aligned} \quad (17)$$

where $\Gamma_Y \triangleq Cov[(Y_0, Y_1, ..., Y_{T-1})^{tr}]$.
Note that in Lemma 3.1, the required stationary condition can be relaxed as long as Λ_∞ or $\lim_{T\to\infty}\frac{1}{T}\ln\det\Gamma_Y$ exist and they finite.

Lemma 3.2: [7] For the nominal source model (16),

$$\Lambda_\infty = CV_\infty C^{tr} + DD^{tr}, \quad (18)$$

where V_∞ is unique positive semi-definite solution of the following Algebraic Riccati-equation

$$\begin{aligned}V_\infty &= AV_\infty A^{tr} - AV_\infty C^{tr}[CV_\infty C^{tr} + DD^{tr}]^{-1} \\ &\quad .CV_\infty A^{tr} + BB^{tr}. \end{aligned}\quad (19)$$

Next, in the following Proposition, using Example 2.8, Lemma 3.1 and Lemma 3.2, we calculate the robust entropy rate for the family of uncertain sources which corresponds to the nominal source model (16) and the relative entropy uncertainty set (15).

Proposition 3.3: The robust entropy rate of an uncertain source with corresponding nominal source model (16) is

$$\begin{aligned}\mathcal{H}_{robust}(\mathcal{Y}) &= \frac{d}{2}\ln(\frac{1+s}{s}) + \mathcal{H}_S(\mathcal{Y}), \\ \mathcal{H}_S(\mathcal{Y}) &\triangleq \frac{d}{2}\ln(2\pi e) + \frac{1}{2}\ln\det\Lambda_\infty, \end{aligned}\quad (20)$$

where $s > 0$ is the unique solution of (14), Λ_∞ is given by (18), and $\mathcal{H}_S(\mathcal{Y})$ is the Shannon entropy rate of the nominal source model (16).

Remark 3.4: From (14), it follows that, the case $R_c \to 0$ corresponds to $s \to +\infty$. Letting $s \to +\infty$ in (20), we obtain

$$\mathcal{H}_{robust}(\mathcal{Y}) = \mathcal{H}_S(\mathcal{Y}). \quad (21)$$

That is, the robust entropy rate is equal to the Shannon entropy rate of the nominal source. This is the result that we expected, since the case $R_c \to 0$ corresponds to the case without uncertainty.

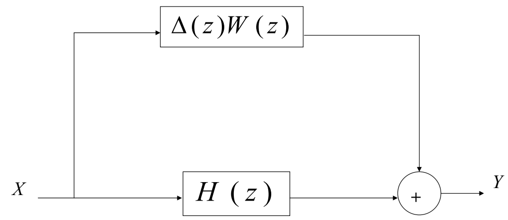

Fig. 1. Source with additive uncertainty

Corollary 3.5: [4] For the scalar case with $B = 0$, after solving V_∞ from (19) we obtain

$$\begin{aligned}\mathcal{H}_{robust}(\mathcal{Y}) &= \frac{1}{2}\ln(\frac{1+s}{s}) + \frac{1}{2}\ln(2\pi e D^2) \\ &\quad + \max\{0, \ln|A|\}. \end{aligned}\quad (22)$$

B. Uncertain Sources in Frequency Domain

Let $\beta(1) \triangleq \{z; z \in \mathcal{C}, |z| \leq 1\}$ and H^∞ be the space of scalar bounded, analytic functions of $z \in \beta(1)$. This space endowed with the norm $||.||_\infty$ defined by $||H(e^{jw})||_\infty \triangleq \sup_{-\pi\leq w\leq\pi}|H(e^{jw})|, (z = e^{jw})$ is a Banach space. Suppose the uncertain source is obtained by passing a stationary Gaussian random process $X : \Omega \times \mathcal{N}_+ \to \Re$, with known power spectral density $S_X(e^{jw})$, through an uncertain linear filter $\tilde{H}(z)$. $\tilde{H}(z)$ belongs to the additive uncertainty model (See Fig. 1)

$$\begin{aligned}&\tilde{H} \in \mathcal{D}_{ad} \triangleq \\ &\Big\{\tilde{H} \in H^\infty; \tilde{H}(z) = H(z) + \Delta(z)W(z); \tilde{H}(z), \\ &H(z), \Delta(z), W(z) \in H^\infty, H(z), W(z) \text{ are fixed}, \\ &\Delta(z) \text{ is unknown and } ||\Delta||_\infty \leq 1\Big\}, \end{aligned}\quad (23)$$

where $H(z)$ is the nominal source transfer function based on previous experience or belief, and $\Delta(z)W(z)$ represents the uncertainty part of the source. Clearly, this additive uncertainty model implies $|\tilde{H}(e^{jw}) - H(e^{jw})| \leq |W(e^{jw})|$, $\forall w \in [-\pi, \pi]$ and thus the size of uncertainty is controlled by the fixed transfer function $W(z)$.
Since X is a Gaussian random process and $\tilde{H}(z)$ is linear transformation, Y is a Gaussian random process. Moreover, since X is stationary and $\tilde{H}(z) \in H^\infty$, $S_Y(e^{jw}) = |\tilde{H}(e^{jw})|^2 S_X(e^{jw})$, consequently from [7], the entropy rate is given by

$$\mathcal{H}_S(\mathcal{Y}) = \frac{1}{2}\ln(2\pi e) + \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln S_Y(e^{jw})dw. \quad (24)$$

Consequently, the robust entropy rate is defined by

$$\begin{aligned}&\mathcal{H}_{robust}(\mathcal{Y}) \triangleq \frac{1}{2}\ln(2\pi e) \\ &+ \frac{1}{4\pi}\sup_{\tilde{H}\in\mathcal{D}_{ad}}\int_{-\pi}^{\pi}\ln S_Y(e^{jw})dw. \end{aligned}\quad (25)$$

Next, from analysis done in [4], the solution of (25) is given by

$$\mathcal{H}_{robust}(\mathcal{Y}) = \frac{1}{2}\ln(2\pi e) + \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln(|H(e^{jw})|^2 + |W(e^{jw})|^2)S_X(e^{jw})dw. \quad (26)$$

C. Partially Observed Controlled Gauss Markov Process

In this section, it is assumed that the uncertainty set is the relative entropy set (15). The nominal model is defined via a partially observed controlled Gauss Markov source given by

$$\begin{aligned} X_{t+1} &= AX_t + BU_t \quad U_t = -KY_t \\ Y_t &= CX_t + DV_t, \quad t \in \mathcal{N}_+, \end{aligned} \quad (27)$$

where K is stabilizing matrix (e.g., $A-BK$ has eigenvalues within the unit circle), $X_t \in \Re^n$ denotes the unobserved process, $Y_t \in \Re$ is the observed process, $U_t \in \Re$, $V_t \in \Re$, V_t is i.i.d. $\sim N(0,1)$, $X_0 \sim N(\bar{x}_0, V_0)$, $\{X_0, V_t\}$ are mutually independent, $t \in \mathcal{N}_+$, and $D \neq 0$. Next, in the following Proposition, using Example 2.8 and the Body integral formula [8], we calculate the robust entropy rate.

Proposition 3.6: The robust entropy rate of an uncertain source with corresponding nominal source model (27) is

$$\begin{aligned} \mathcal{H}_{robust}(\mathcal{Y}) &= \frac{1}{2}\ln(\frac{1+s}{s}) + \mathcal{H}_S(\mathcal{Y}) \\ \mathcal{H}_S(\mathcal{Y}) &\triangleq \frac{1}{2}\ln(2\pi e D^2) \\ &+ \sum_{\{i:|\lambda_i(A)|\geq 1\}} \ln|\lambda_i(A)|, \end{aligned} \quad (28)$$

where $s > 0$ is the unique solution of (14), $\mathcal{H}_S(\mathcal{Y})$ is the Shannon entropy rate of the nominal source model (27), and $\lambda_i(A)$ is the eigenvalues of the system matrix A.

IV. Application in Stabilizability of Network Control System

An application of information theory in networked control systems (See Fig. 2), subject to uncertainty in the source requires that the robust channel capacity of the uncertain communication link must be at least equal to the robust entropy rate of the family of uncertain sources [9], in order to have uniform asymptotic observability and stabilizability in probability as defined below.

Definition 4.1: The uncertain source is uniform asymptotic observable in probability if there exists an encoder and decoder such that

$$\lim_{t\to+\infty} \sup_{f\in\mathcal{D}_{SU}} \frac{1}{t}\sum_{k=0}^{t-1} \Pr(||Y_k - \widetilde{Y}_k||_2 > \delta) \leq \epsilon, \quad (29)$$

where $\delta \geq 0$ and $0 \leq \epsilon \leq 1$ are fixed and $||y||_2 = (y^{tr}y)^{\frac{1}{2}}$. The uncertain source is uniform asymptotic stabilizable in probability if there exists encoder, decoder, and controller such that

$$\lim_{t\to\infty} \sup_{f\in\mathcal{D}_{SU}} \frac{1}{t}\sum_{k=0}^{t-1} \Pr(||Y_k||_2 > \delta) \leq \epsilon. \quad (30)$$

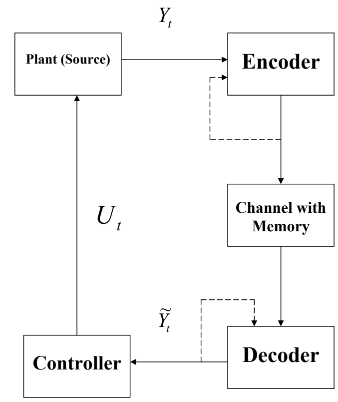

Fig. 2. Networked control system

Next, we have the following proposition which gives necessary condition for uniform asymptotic observability and stabilizability in terms of the robust entropy rate.

Proposition 4.2: [9] A necessary condition on the robust channel capacity ($C_{robust} = \lim_{n\to\infty}\frac{1}{n}C_{n,robust}$, where $C_{n,robust}$ is the robust channel capacity for n-times channel use) for uniform asymptotic observability and stabilizability in probability is

$$C_{robust} \geq \mathcal{H}_{robust}(\mathcal{Y}) - \frac{1}{2}\ln(2\pi e)^d \det\Gamma_g, \quad (31)$$

where $\mathcal{H}_{robust}(\mathcal{Y})$ is the robust entropy rate, d is the dimension of source symbol, and Γ_g is the covariance matrix of the Gaussian distribution $g(y) \sim N(0, \Gamma_g)$ ($y \in \Re^d$) that satisfies $\int_{||y||_2>\delta} g(y)dy = \epsilon$.

Remark 4.3: For the family of uncertain sources that corresponds to the nominal source model (16), $\mathcal{H}_{robust}(\mathcal{Y})$ (found in (31)) is computed from (20). For an uncertain sources given in Section III-B ($\mathcal{D}_{SU} \to \mathcal{D}_{ad}$), it is found from (26).

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