Stability Of Nonlinear Uncertain Lipschitz Systems Over The Digital Noiseless Channel

Alireza Farhadi

Abstract

This paper is concerned with the stability of nonlinear Lipschitz systems subject to bounded process and measurement noises when transmission from sensor to controller is subject to distortion due to quantization. A stabilizing technique and a sufficient condition relating transmission rate to Lipschitz coefficients are presented for almost sure asymptotic bounded stability of nonlinear uncertain Lipschitz systems. In the absence of process and measurement noises, it is shown that the proposed stabilizing technique results in almost sure asymptotic stability. Computer simulations illustrate the satisfactory performance of the proposed technique for almost sure asymptotic bounded stability and asymptotic stability.

Index Terms

Networked control system, Lipschitz nonlinear system, uncertain dynamic system, the digital noiseless channel.

I. Introduction

A. Motivation and Background

Recently, stabilizing a dynamic system over a communication channel subject to imperfections (e.g., quantization distortion) has became an important problem. Some examples of systems that are required to be stabilized over communication channels subject to imperfections are smart drilling system using borehole telemetry via drilling string [1],[2] and distributed monitoring system of oil fields. In these systems, transmission between sub-components (e.g., sensor, controller,

A. Farhadi is an Assistant Professor in the Department of Electrical Engineering, Sharif University of Technology, Tehran, Iran, email: afarhadi@sharif.edu.

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actuator) is subject to imperfections, such as quantization distortion. Some results addressing basic problems in state estimation and/or stability of dynamic systems over communication channels subject to imperfections can be found in [3]-[17]. In [15] the authors addressed the problem of state estimation of an uncontrolled noiseless nonlinear Lipschitz system over the digital noiseless channel with asymptotically zero mean square estimation error. [16] addressed the problem of state estimation of distributed uncontrolled Lipschitz systems subject to bounded process and measurement noises over the packet erasure network with bounded mean absolute estimation error; and [12] addressed the stability problem of nonlinear noiseless systems over the digital noiseless channel.

This paper is concerned with a basic problem in the stability of nonlinear dynamic systems subject to uncertain transmission as described in Fig. 1. The block diagram of Fig. 1 has been considered in several research papers addressing basic problems in networked control systems, such as [5], [12]. In this paper, the system shown in Fig. 1 is described by a nonlinear controlled Lipschitz system subject to bounded process and measurement noises over the digital noiseless channel. A large class of nonlinear systems, such as systems with dead zone and saturation nonlinearities are Lipschitz. Furthermore, important class of linear systems is a special class of Lipschitz systems. The digital noiseless channel is also a basic digital communication channel.

B. *Paper Contributions*

In this paper, we address the problem of almost sure bounded stability of controlled nonlinear Lipschitz systems subject to bounded process and measurement noises when measurements from dynamic system sampled by sensor are transmitted via the digital noiseless channel to controller (see Fig. 1). As the sampled measurements are real valued, to transmit them over digital links, they must be quantized and represented as a packet of binary data with a specific length (e.g., R bits). This results in distortion in sampled measurements when they are reconstructed at controller. That is, another source of uncertainty considered in this paper is distortion in measurements due to quantization. Despite of these uncertainties, a stabilizing technique (which consists of an encoder, decoder and a controller) and a sufficient condition relating transmission rate R to Lipschitz coefficients are presented that result in almost sure asymptotic bounded stability. In the absence of process and measurement noises, it is shown that the proposed stabilizing technique results in almost sure asymptotic stability despite of distortion in measurements due to quantization. The satisfactory performance of the proposed stabilizing technique for almost sure asymptotic bounded stability and asymptotic stability is also illustrated via computer simulations.

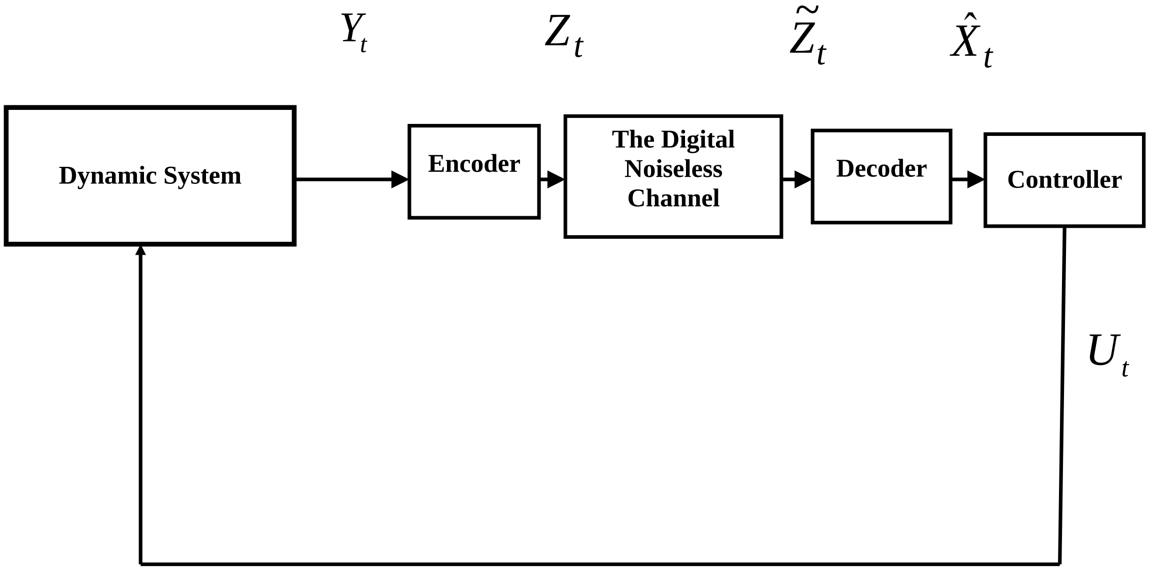

Fig. 1. A control system over the digital noiseless channel.

To the best of our knowledge, only the problem of the state estimation of nonlinear uncontrolled Lipschitz dynamic systems over communication channels subject to imperfections (known as tracking) has been addressed in the literature; and hence, the main novelty of this paper is the stability of nonlinear noisy controlled Lipschitz systems, in addition of tracking, over the digital noiseless channel. The problem of the state estimation of uncontrolled nonlinear Lipschitz systems over digital links was first considered in [15]. In [15] the authors addressed the problem of the state estimation of a noiseless uncontrolled Lipschitz system over the digital noiseless channel subject to quantization effect. In [16] this result was extended to distributed uncontrolled noisy Lipschitz systems over the packet erasure network.

C. Paper Organization

The paper is organized as follows: In Section II, the problem formulation is given. Section III is devoted to the stability result. In this section, encoder, decoder, controller and a sufficient condition for almost sure asymptotic bounded stability are presented. Simulation results are given in Section IV; and the paper is concluded in Section V by summarizing the main contributions of the paper.

II. Problem Formulation

Throughout, certain conventions are used: $|\cdot|$ denotes the absolute value, $\log_2(\cdot)$ denotes the logarithm of base 2 and '$\doteq$' means 'by definition is equivalent to'. $[X]_i$ means the i-th element of the vector X and $\mathbb{R}$ denotes the set of real numbers. Cartesian product is denoted by $\times$ and

A' denotes the transpose of the matrix A. A^{-1} denotes the inverse of the square matrix A.

This paper is concerned with almost sure asymptotic bounded stability of nonlinear Lipschitz dynamic systems over the digital noiseless communication channel, as shown in the block diagram of Fig. 1. The building blocks of Fig. 1 are described below.

Dynamic System: The dynamic system is described by the following discrete time nonlinear Lipschitz system

$$\begin{cases} X_{t+1} = F(X_t) + BU_t + W_t, \quad X_0 = \xi, \\ \quad Y_t = X_t + V_t, \end{cases}, \quad t \in \mathbb{N}_+ \doteq \{0, 1, 2, ...\}, \tag{1}$$

where $X_t \in \mathbb{R}^n$ is the state of the system, n is the number of state variables and it is given, Y_t is the observation signal, the random variable ξ is the unknown initial state value, $U_t \in \mathbb{R}^m$ is the control vector, $F(\cdot) \in \mathbb{R}^n$ is a nonlinear continuous function with components $f_i(\cdot)$, and $B \in \mathbb{R}^{n \times m}$ is such that the matrix BB' is invertible. Throughout, it is assumed that the probability measure associated with the initial state X_0 with components $X_0^{(i)}$, $i = \{1, 2, ..., n\}$, has bounded support. That is, for each $i = \{1, 2, ..., n\}$ there exists a compact set $[-L_0^{(i)}, L_0^{(i)}] \in \mathbb{R}$ such that $\Pr(X_0^{(i)} \in [-L_0^{(i)}, L_0^{(i)}]) = 1$. Also, for each i, $f_i(\cdot)$ is Lipschitz. That is, for each $f_i(\cdot)$ there exists a $K_i > 0$ (known as Lipschitz coefficient) such that the following inequality holds for all $X = \begin{pmatrix} X^{(1)} & X^{(2)} & ... & X^{(n)} \end{pmatrix}' \in \mathbb{R}^n, Y = \begin{pmatrix} Y^{(1)} & Y^{(2)} & ... & Y^{(n)} \end{pmatrix}' \in \mathbb{R}^n$

$$|f_i(X) - f_i(Y)| \leq K_i(|X^{(1)} - Y^{(1)}| + |X^{(2)} - Y^{(2)}| + ... + |X^{(n)} - Y^{(n)}|).$$

In the dynamic system (1), $W_t \in \mathbb{R}^n$ with the components $W_t^{(i)}$, $i = \{1, 2, ..., n\}$, is the process noise vector and $V_t \in \mathbb{R}^n$ with the components $V_t^{(i)}$, $i = \{1, 2, ..., n\}$, is the measurement noise vector. Throughout, it is assumed that $W_t^{(i)}$ and $V_t^{(i)}$ are uniformly distributed random variables with supports $[-\omega^{(i)}, \omega^{(i)}]$ and $[-\gamma^{(i)}, \gamma^{(i)}]$, respectively (i.e., $W_t^{(i)} \in [-\omega^{(i)}, \omega^{(i)}]$ and $V_t^{(i)} \in [-\gamma^{(i)}, \gamma^{(i)}]$).

Communication Channel: Communication channel between system and controller is the digital noiseless channel with transmission rate R bits. This channel transmits a packet of binary data with rate R bits in each channel use.

As the sampled measurements are real valued, to transmit them over the digital noiseless channel, they must be quantized and represented as a packet of binary data with length R bits. This is done by the encoder in the block diagram of Fig. 1. On the other hand, the decoder reconstructs the quantized sampled measurements at the receiver. The formal description of the encoder and decoder is given below.

Encoder: Encoder is a causal operator denoted by $Z_t = \mathcal{E}(Y_t)$ that maps the system output Y_t to channel input Z_t, which is a string of binaries with length R bits.

Decoder: Decoder is a causal operator denoted by $\hat{X}_t = \mathcal{D}(\tilde{Z}_t)$ that maps the channel output to $\hat{X}_t$ (the estimate of the state variable at the decoder).

Controller: Controller has the following form $U_t = -B'(BB')^{-1}F(\hat{X}_t)$.

The objective of this paper is to design an encoder, decoder and a controller that result in almost sure asymptotic bounded stability of the system (1), as defined below.

Definition 2.1: (Almost Sure Asymptotic Bounded Stability). Consider the block diagram of Fig. 1 described by the nonlinear dynamic system (1) over the digital noiseless channel, as described above. It is said that the system is almost sure asymptotic bounded stabilizable if there exist an encoder, decoder, controller and a closed bounded set $\Delta \subset \mathbb{R}^n$ such that the following property holds:

$$\Pr(\lim_{t\to\infty} X_t \in \Delta) = 1.$$

Remark 2.2: For $\Delta = \{0\}$ we have almost sure asymptotic stability.

III. Stability Result

In this section, we present a sufficient condition on the transmission rate R such that using the controller $U_t = -B'(BB')^{-1}F(\hat{X}_t)$, the dynamic system (1) in the block diagram of Fig. 1 is almost sure asymptotic bounded stable. This result is shown in Theorem 3.1. To obtain this sufficient condition in Theorem 3.1 we construct an encoder and a decoder so that under this condition, the controller $U_t = -B'(BB')^{-1}F(\hat{X}_t)$ results in almost sure asymptotic bounded stability.

For simplicity in understanding these encoder and decoder functions, suppose the system (1) is scalar (i.e., $n = 1$). At time instant $t = 0$, both encoder and decoder divide the interval $[-L_0^{(1)} - \gamma^{(1)}, L_0^{(1)} + \gamma^{(1)}]$ ($Y_0^{(1)} \in [-L_0^{(1)} - \gamma^{(1)}, L_0^{(1)} + \gamma^{(1)}]$) into 2^R equal size non-overlapping sub-intervals and choose the center of each sub-interval as the index of the sub-interval. Now, upon observing $Y_0^{(1)}$, encoder determines the sub-interval where $Y_0^{(1)}$ is located and represents the index of this sub-interval (denoted by j) by R bits; and transmit this R bits to decoder via the digital noiseless channel. Decoder after receiving this R bits determines the index of the sub-interval where $Y_0^{(1)}$ is located (i.e., j) and outputs $\hat{Y}_0^{(1)}$ ($= \hat{X}_0^{(1)}$)$= j$, where $\hat{Y}_0^{(1)}$ is the estimation of $Y_0^{(1)}$ and $\hat{X}_0^{(1)}$ is the estimation of $X_0^{(1)}$ at the end of communication link. Hence, the decoding error for this case is bounded above by $|Y_0^{(1)} - \hat{Y}_0^{(1)}| \leq \frac{L_0^{(1)}+\gamma^{(1)}}{2^R}$.

At time instant $t=1$, encoder computes the error $Y_1^{(1)}-\tilde{Y}_1^{(1)}$, where $\tilde{Y}_1^{(1)}=f_1(\hat{X}_0^{(1)})+BU_0$ ($U_0=-\frac{1}{B}f_1(\hat{X}_0^{(1)})$); and both encoder and decoder divide the interval $Y_1^{(1)}-\tilde{Y}_1^{(1)}\in[-L_1^{(1)}-\gamma^{(1)},L_1^{(1)}+\gamma^{(1)}]$, where $L_1^{(1)}=K_1(\frac{L_0^{(1)}+\gamma^{(1)}}{2^R}+\gamma^{(1)})+\omega^{(1)}$ into 2^R sub-intervals; and they repeat the above procedure until decoder outputs $\hat{Y}_1^{(1)}(=\hat{X}_1^{(1)})=j+\tilde{Y}_1^{(1)}$. As a result, the decoding error is bounded above by $|Y_1^{(1)}-\hat{Y}_1^{(1)}|\leq\frac{L_1^{(1)}+\gamma^{(1)}}{2^R}$.

By following the above procedure, $\hat{X}_0^{(1)}$, $\hat{X}_1^{(1)}$, $\hat{X}_2^{(1)}$, ... are constructed; and decoding error is bounded above by $|Y_t^{(1)}-\hat{Y}_t^{(1)}|\leq\frac{L_t^{(1)}+\gamma^{(1)}}{2^R}$, where

$$L_t^{(1)}=\frac{K_1}{2^R}L_{t-1}^{(1)}+\frac{K_1\gamma^{(1)}}{2^R}+K_1\gamma^{(1)}+\omega^{(1)}.$$

Now, under the assumption of $|\frac{K_1}{2^R}|<1$, the above dynamic system for $L_t^{(1)}$ is stable; and hence, $L_t^{(1)}$ is asymptotically bounded. Therefore, under this assumption, using the above encoding and decoding technique, tracking of $Y_t^{(1)}$ by $\hat{Y}_t^{(1)}$ with bounded error at the end of communication is achieved.

Now, after this instruction and simple explanation of the designed method and algorithm, we are ready to present the main theorem of this paper.

Theorem 3.1: Consider the control system of Fig. 1 described by the nonlinear uncertain Lipschitz system (1) over the digital noiseless channel, as described earlier. Suppose that there exists non-negative integers R_1, R_2, ... , R_n that make the following matrix stable:

$$\mathcal{A}\doteq\begin{pmatrix} \frac{K_1}{2^{R_1}} & \frac{K_1}{2^{R_2}} & . & . & . & \frac{K_1}{2^{R_n}} \\ \frac{K_2}{2^{R_1}} & \frac{K_2}{2^{R_2}} & . & . & . & \frac{K_2}{2^{R_n}} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \frac{K_n}{2^{R_1}} & \frac{K_n}{2^{R_2}} & . & . & . & \frac{K_n}{2^{R_n}} \end{pmatrix}. \tag{2}$$

Then, using the control policy $U_t=-B'(BB')^{-1}F(\hat{X}_t)$, there exists a closed bounded set $\Delta\subset\mathbb{R}^n$ such that $X_t\to\Delta$, P-a.s.; or equivalently, $\Pr(\lim_{t\to\infty}X_t\in\Delta)=1$, where $\Delta=[-L_\infty^{(1)},L_\infty^{(1)}]\times[-L_\infty^{(2)},L_\infty^{(2)}]\times\cdots\times[-L_\infty^{(n)},L_\infty^{(n)}]$, $L_\infty^{(i)}\doteq[\lim_{t\to\infty}\sum_{j=0}^{t-1}\mathcal{A}^{t-1-j}((\mathcal{A}+\mathcal{B})\gamma+\omega)]_i$.

Proof: To prove this theorem we show that the extended version of the above encoding and decoding technique applied to the system (1) with n states along with the controller $U_t=-B'(BB')^{-1}F(\hat{X}_t)$ result in almost sure asymptotic bounded stability if the matrix (2) is stable.

Encoding and Decoding Technique: At time instant $t=0$ for each $i=\{1,2,...,n\}$ the set $[-L_0^{(i)}-\gamma^{(i)},L_0^{(i)}+\gamma^{(i)}]$ is partitioned into 2^{R_i} equal size, non-overlapping sub-intervals and the

center of each sub-interval is chosen as the index of that interval. For each $i = \{1, 2, ..., n\}$ upon observing $Y_0(= X_0 + V_0)$, the index of the sub-interval that includes $Y_0^{(i)}$ is encoded into R_i bits. Then, a packet with length $R = R_1 + R_2 + ... + R_n$ bits containing information about the initial measurement Y_0 is transmitted via the channel. When the decoder receives these R bits, for each i it identifies the index of the sub-interval where $Y_0^{(i)}$ is located; and the value of this index is chosen as $\hat{Y}_0^{(i)} = \hat{X}_0^{(i)}$ (the estimation of $Y_0^{(i)}$ and $X_0^{(i)}$ at the receiver). Therefore, the estimation error is bounded above by $|X_0^{(i)} - \hat{X}_0^{(i)}| \leq \frac{L_0^{(i)} + \gamma^{(i)}}{2^{R_i}} + \gamma^{(i)}$.
At time instant $t = 1$, from the Lipschitz continuity assumption, for each $i = \{1, 2, ..., n\}$, an upper bound on $X_1^{(i)}$ is calculated as follows

$$\begin{aligned} |X_1^{(i)}| &= |f_i(X_0) + [BU_0]_i + W_0^{(i)}| = |f_i(X_0) - f_i(\hat{X}_0) + W_0^{(i)}| \\ &\leq K_i(|X_0^{(1)} - \hat{X}_0^{(1)}| + ... + |X_0^{(n)} - \hat{X}_0^{(n)}|) + \omega^{(i)} = \omega^{(i)} + K_i \sum_{j=1}^{n} (\frac{L_0^{(j)} + \gamma^{(j)}}{2^{R_j}} + \gamma^{(j)}) \doteq L_1^{(i)}. \end{aligned}$$

Then, similar to the previous time instant, at this time instant, for each $i = \{1, 2, ..., n\}$, the interval $[-L_1^{(i)} - \gamma^{(i)}, L_1^{(i)} + \gamma^{(i)}]$ is partitioned into 2^{R_i} equal size, non-overlapping sub-intervals and the center of each sub-interval is chosen as the index of that interval. Having that, for each i, upon observing $Y_1(= X_1 + V_1)$, the index of the sub-interval that includes $Y_1^{(i)}$ is encoded into R_i bits. Then, $R = R_1 + R_2 + ... + R_n$ bits containing information about Y_1 is transmitted via the channel. When these R bits are received, for each i the decoder identifies the index of the sub-interval that contains $Y_1^{(i)}$; and the value of this index is chosen as $\hat{Y}_1^{(i)} = \hat{X}_1^{(i)}$. Therefore, the estimation error is bounded above by $|X_1^{(i)} - \hat{X}_1^{(i)}| \leq \frac{L_1^{(i)} + \gamma^{(i)}}{2^{R_i}} + \gamma^{(i)}$.
By following a similar procedure, as described above, the sequence $\hat{X}_0$, $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$, ..., are constructed at the decoder.

Now, we must show that using the above coding technique and controller $U_t = -B'(BB')^{-1}F(\hat{X}_t)$, there exists a closed and bounded set $\Delta \subset \mathbb{R}^n$ such that $\Pr(\lim_{t\to\infty} X_t \in \Delta) = 1$ provided the matrix (2) is stable. To achieve this goal, choose any rates R_1, R_2, ..., R_n that make the matrix $\mathcal{A}$ stable. Now, using the above encoding and decoding technique and the controller $U_t = -B'(BB')^{-1}F(\hat{X}_t)$, we have

$$\begin{aligned} |X_0^{(i)}| &\leq L_0^{(i)} \\ |X_1^{(i)}| &\leq \omega^{(i)} + K_i \sum_{j=1}^{n} (\frac{L_0^{(j)} + \gamma^{(j)}}{2^{R_j}} + \gamma^{(j)}) \doteq L_1^{(i)} \\ |X_2^{(i)}| &\leq \omega^{(i)} + K_i \sum_{j=1}^{n} (\frac{L_1^{(j)} + \gamma^{(j)}}{2^{R_j}} + \gamma^{(j)}) \doteq L_2^{(i)} \end{aligned}$$

$$
\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
|X_t^{(i)}| \;\; \leq \;\; \omega^{(i)} + K_i \sum_{j=1}^{n} (\frac{L_{t-1}^{(j)} + \gamma^{(j)}}{2^{R_j}} + \gamma^{(j)}) \doteq L_t^{(i)}.
\end{array}
$$

Now, let

$$
Z_t \doteq \begin{pmatrix} L_t^{(1)} \\ L_t^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ L_t^{(n)} \end{pmatrix}.
$$

Then, from the recursive equation $L_t^{(i)} = \omega^{(i)} + K_i \sum_{j=1}^{n} (\frac{L_{t-1}^{(j)} + \gamma^{(j)}}{2^{R_j}} + \gamma^{(j)})$, which defines, almost surely, an upper bound on $X_t^{(i)}$, we have the following dynamic model for the vector Z_t:

$$
Z_{t+1} = \mathcal{A} Z_t + (\mathcal{A} + \mathcal{B})\gamma + \omega, \tag{3}
$$

where

$$
Z_0 = \begin{pmatrix} L_0^{(1)} \\ L_0^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ L_0^{(n)} \end{pmatrix}, \quad \mathcal{B} \doteq \begin{pmatrix} K_1 & K_1 & . & . & . & K_1 \\ K_2 & K_2 & . & . & . & K_2 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ K_n & K_n & . & . & . & K_n \end{pmatrix}, \quad \gamma \doteq \begin{pmatrix} \gamma^{(1)} \\ \gamma^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \gamma^{(n)} \end{pmatrix}, \quad \omega \doteq \begin{pmatrix} \omega^{(1)} \\ \omega^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \omega^{(n)} \end{pmatrix}.
$$

Now, from the well known stability results of linear time-invariant systems, for the linear time-invariant dynamic system (3), it follows that all components of the vector Z_t are asymptotically bounded if and only if the matrix $\mathcal{A}$ is stable. Hence, as we chose the rates R_1, R_2, ..., R_n such that the matrix $\mathcal{A}$ is stable and as $Z_t^{(i)}$ defines, almost surely, an upper bound on $X_t^{(i)}$, the dynamic system (1) is almost sure asymptotic bounded stable as follows: $X_t \to \Delta$, P-a.s., where $\Delta = [-L_\infty^{(1)}, L_\infty^{(1)}] \times [-L_\infty^{(2)}, L_\infty^{(2)}] \times \cdots \times [-L_\infty^{(n)}, L_\infty^{(n)}]$, $L_\infty^{(i)} = [\lim_{t\to\infty} \sum_{j=0}^{t-1} \mathcal{A}^{t-1-j}((\mathcal{A}+\mathcal{B})\gamma+\omega)]_i$. We have the following two propositions as a direct result of the above theorem.

Proposition 3.2: For the noiseless dynamic system, i.e., $\gamma^{(i)} = \omega^{(i)} = 0$, we have almost sure asymptotic stability as follows $X_t \to 0$, P-a.s., provided the rates R_1, R_2, ..., R_n make the matrix $\mathcal{A}$ stable.

Proof: As the rates R_1, R_2, ..., R_n make the matrix $\mathcal{A}$ stable, from the Theorem 3.1 it follows that $X_t \to \Delta$, P-a.s., where $\Delta = [-L_\infty^{(1)}, L_\infty^{(1)}] \times [-L_\infty^{(2)}, L_\infty^{(2)}] \times \cdots \times [-L_\infty^{(n)}, L_\infty^{(n)}]$, $L_\infty^{(i)}$=$[\lim_{t\to\infty} \sum_{j=0}^{t-1} \mathcal{A}^{t-1-j}((\mathcal{A}+\mathcal{B})\gamma + \omega)]_i$. Now, as for each $i = \{1, 2, ..., n\}$, it is assumed that $\gamma^{(i)} = \omega^{(i)} = 0$, we have $\sum_{j=0}^{t-1} \mathcal{A}^{t-1-j}((\mathcal{A}+\mathcal{B})\gamma + \omega) = 0$; and hence, $X_t \to 0$, P-a.s.

Proposition 3.3: Consider the control system of Fig. 1 described by the scalar version of the nonlinear uncertain Lipschitz system (1) over the digital noiseless channel with rate $R > \log_2 K_1$, where $K_1 > 0$ is the Lipschitz coefficient (i.e., $|f_1(X) - f_1(Y)| \leq K_1|X - Y|$, $\forall X, Y \in \mathbb{R}$). Then, using the proposed encoding and decoding technique and $U_t = -\frac{1}{B} f_1(\hat{X}_t^{(1)})$, there exists the set $\Delta = [-((\frac{K_1}{2^R} + K_1)\gamma^{(1)} + \omega^{(1)})\frac{1}{1-\frac{K_1}{2^R}}, ((\frac{K_1}{2^R} + K_1)\gamma^{(1)} + \omega^{(1)})\frac{1}{1-\frac{K_1}{2^R}}]$ such that $X_t^{(1)} \to \Delta$, P-a.s.; or equivalently, $\Pr(\lim_{t\to\infty} X_t^{(1)} \in \Delta) = 1$.

Proof: For the scalar system, the matrix $\mathcal{A}$ is reduced to $\mathcal{A} = (\ \frac{K_1}{2^R}\ )$, which is stable for any rate $R > \log_2 K_1$. Hence, as we assumed that $R > \log_2 K_1$, it follows from the Theorem 3.1 that $X_t \to \Delta$, P-a.s., where $\Delta = [-L_\infty^{(1)}, L_\infty^{(1)}]$, $L_\infty^{(1)} = \lim_{t\to\infty} \sum_{j=0}^{t-1} \mathcal{A}^{t-1-j}((\mathcal{A}+\mathcal{B})\gamma^{(1)} + \omega^{(1)}) = ((\frac{K_1}{2^R} + K_1)\gamma^{(1)} + \omega^{(1)})\frac{1}{1-\frac{K_1}{2^R}}$. This completes the proof.

Remark 3.4: i) From the specific structure of the matrix $\mathcal{A}$ it follows that the eigenvalues of this matrix are: $0, 0, ..., 0, \frac{K_1}{2^{R1}} + \frac{K_2}{2^{R_2}} + ... + \frac{K_n}{2^{R_n}}$. Hence, a sufficient condition on the rates R_is for the stability using the proposed stabilizing technique is the following condition:

$$R_i > \max\{0, \log_2 K_i\}, \quad \forall i \in \{1, 2, ..., n\}. \tag{4}$$

ii) In general, the weaker condition

$$R \geq \sum_{i; K_i > 1} \log_2 K_i \tag{5}$$

does not imply the stronger condition (4). However, for those cases that the weaker condition (5) implies the stronger condition (4) (e.g., this is the case for $K_1 = 5$ and $K_2 = 7$), we can conclude that the condition (5) is also a sufficient condition for stability.

iii) For linear time-invariant noiseless systems with eigenvalues $\lambda_i(A)$s (A is the system matrix) over the packet erasure channel with rate R bits (which includes the digital noiseless channel as a special case), it is shown in [17] that the condition (6) on rates R_1, R_2, ..., R_n ($R = R_1 + R_2 + ... + R_n$) is a sufficient condition for almost sure asymptotic stability

$$R_i > \max\{0, \log_2 |\lambda_i(A)|\}, \quad \forall i \in \{1, 2, ..., n\}. \tag{6}$$

Furthermore, independent of the choice of encoder, decoder and controller, the following condition, known as the eigenvalues rate condition, is a necessary condition for almost sure asymptotic stability:

$$R \geq \sum_{i;|\lambda_i(A)|>1} \log_2 |\lambda_i(A)|. \tag{7}$$

In general, the eigenvalues rate condition does not imply the stronger condition (6). But, for those cases that the eigenvalues rate condition implies the stronger condition (6) (e.g., this is the case for the system matrix $A = \begin{pmatrix} 11 & 8 \\ -3 & 1 \end{pmatrix}$), in [17] it is concluded that the eigenvalues rate condition (7) is a necessary and sufficient condition (a tight bound on transmission rate R) for almost sure asymptotic stability of linear time-invariant noiseless systems over the packet erasure channel.

iv) From the above remarks it follows that for linear time-invariant noiseless systems over the digital noiseless channel, the condition (5) is a necessary and sufficient condition (a tight bound on transmission rate) for almost sure asymptotic stability if $K_i = |\lambda_i(A)|$ and the eigenvalues rate condition implies the stronger condition (6).

To the best of our knowledge, similar works to this work were previously reported in [15] and [16], where they addressed only the problem of tracking states of uncontrolled Lipschitz systems over the digital noiseless and the packet erasure channels, respectively. Note that the digital noiseless channel is a special case of the packet erasure channel when the erasure probability is zero. In [15] the authors considered noiseless uncontrolled Lipschitz systems and presented a sufficient condition for mean square asymptotic tracking, in which for the scalar system this condition is reduced to the condition found in this paper for tracking (i.e., $|\frac{K_1}{2^R}| < 1$). In [16] the authors addressed the problem of tracking of a distributed system of uncontrolled Lipschitz distributed noisy sub-systems over the packet erasure network. For mean absolute tracking, which is a weaker notion for tracking than almost sure notion used in this paper, they found a sufficient condition, which for the special case of single sub-system, is reduced to the condition found in this paper.

IV. Simulation Results

In this section, we illustrate the satisfactory performance of the proposed encoder, decoder and controller for almost sure asymptotic bounded stability and asymptotic stability using computer simulations.

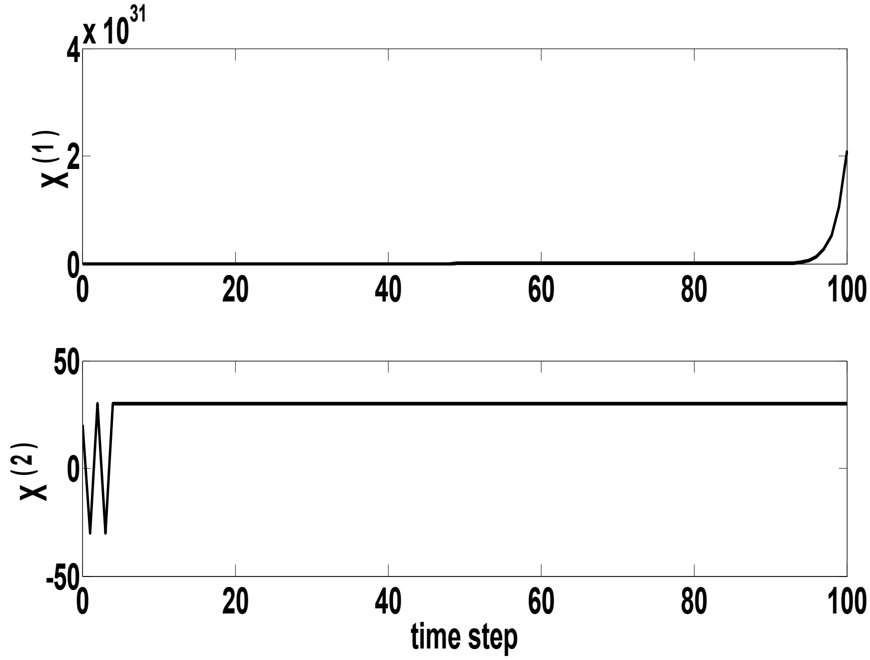

Fig. 2. The state trajectories without control inputs.

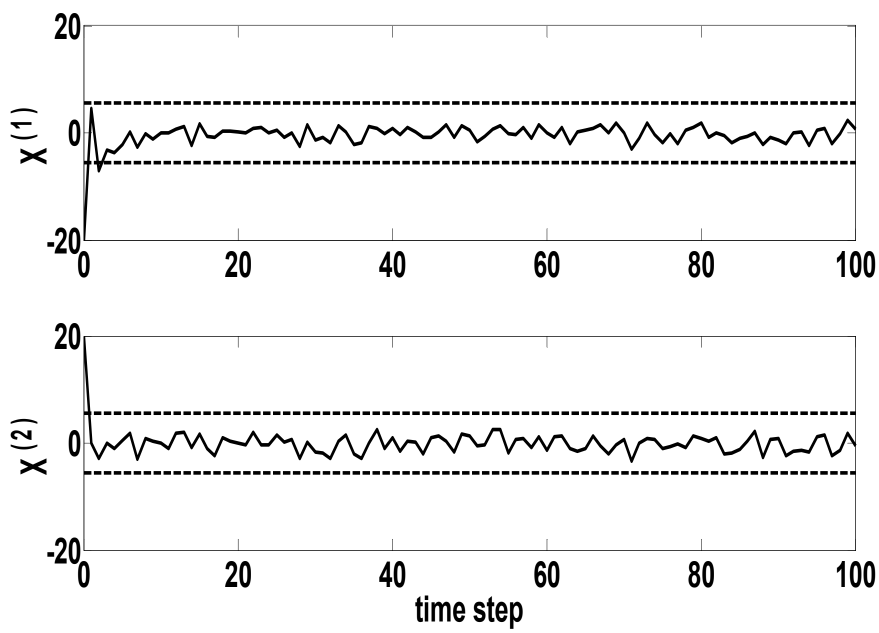

Fig. 3. The state trajectories for (R_1, R_2)= $(4, 4)$. Dashed lines identify the boundaries of the interval $[\frac{-44}{8}, \frac{44}{8}]$.

Define the nonlinear Lipschitz functions $sat(\cdot)$ and $deadz(\cdot)$ as follows:

$$sat(x) \doteq \begin{cases} 30, & x \geq 10 \\ 3x, & -10 < x < 10 \\ -30, & x \leq -10 \end{cases} \qquad deadz(x) \doteq \begin{cases} 2(x-1), & x \geq 1 \\ 0, & -1 < x < 1 \\ 2(x+1), & x \leq -1 \end{cases}$$

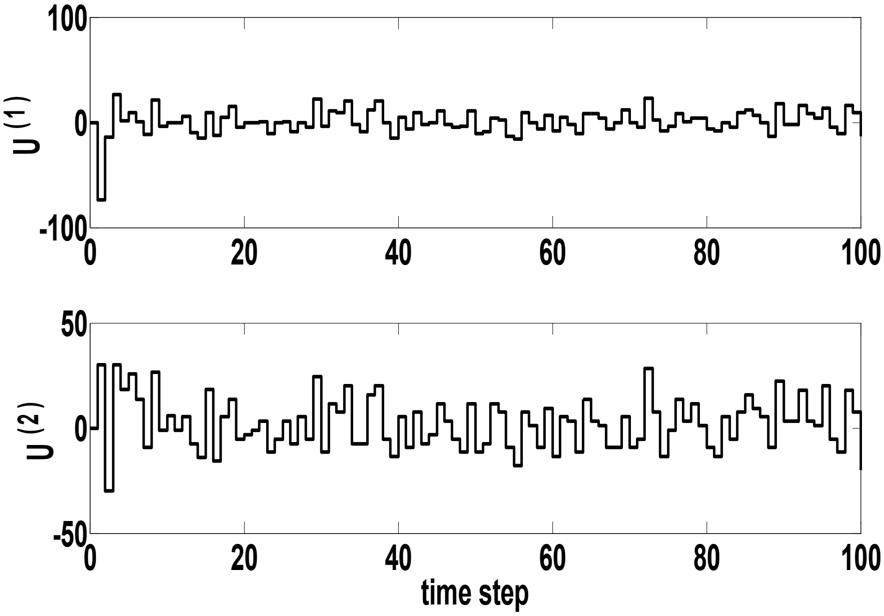

Fig. 4. The control trajectories for (R_1, R_2)= $(4, 4)$.

Now, suppose the control system of Fig. 1 is described by the following coupled nonlinear system.

$$\begin{cases} X_{t+1}^{(1)} = deadz(X_t^{(1)} + 3X_t^{(2)}) + U_t^{(1)} + W_t^{(1)} \\ Y_t^{(1)} = X_t^{(1)} + V_t^{(1)} \\ X_{t+1}^{(2)} = sat(2X_t^{(1)} + X_t^{(2)}) + U_t^{(2)} + W_t^{(2)} \\ Y_t^{(2)} = X_t^{(2)} + V_t^{(2)} \end{cases} \tag{8}$$

Here, $X_0^{(1)}, X_0^{(2)} \in [-20, 20]$ are unknown initial states and $W_t^{(i)}$ and $V_t^{(i)}$ are uniformly distributed random variables with the support $[-0.1, 0.1]$ (i.e., $W_t^{(i)}, V_t^{(i)} \in [-0.1, 0.1]$).

Fig. 2 illustrates the state trajectories of the system (8) when $U_t^{(1)} = U_t^{(2)} = 0$. As is clear from Fig. 2, without control inputs, the system is unstable.

To stabilize this system, the control vector is set to be $U_t = \begin{pmatrix} U_t^{(1)} \\ U_t^{(2)} \end{pmatrix}$, $U_t^{(1)} = -deadz(\hat{X}_t^{(1)} + 3\hat{X}_t^{(2)})$, $U_t^{(2)} = -sat(2\hat{X}_t^{(1)} + \hat{X}_t^{(2)})$. For this system, the Lipschitz coefficients K_1, K_2 are determined as follows:

$$\begin{aligned} |deadz(X^{(1)} + 3X^{(2)}) - deadz(Y^{(1)} + 3Y^{(2)})| &\leq 2|X^{(1)} + 3X^{(2)} - Y^{(1)} - 3Y^{(2)}| \\ &= 2|(X^{(1)} - Y^{(1)}) + 3(X^{(2)} - Y^{(2)})| \\ &\leq 2|X^{(1)} - Y^{(1)}| + 6|X^{(2)} - Y^{(2)}| \end{aligned}$$

$$\leq \quad 6(|X^{(1)} - Y^{(1)}| + |X^{(2)} - Y^{(2)}|).$$

Hence, $K_1 = 6$. Note that the first inequality above follows from the definition of $deadz(.)$. For K_2 similarly, we have:

$$\begin{aligned} |sat(2X^{(1)} + X^{(2)}) - sat(2Y^{(1)} + Y^{(2)})| &\leq 3|2X^{(1)} + X^{(2)} - 2Y^{(1)} - Y^{(2)}| \\ &= 3|2(X^{(1)} - Y^{(1)}) + (X^{(2)} - Y^{(2)})| \\ &\leq 6|X^{(1)} - Y^{(1)}| + 3|X^{(2)} - Y^{(2)}| \\ &\leq 6(|X^{(1)} - Y^{(1)}| + |X^{(2)} - Y^{(2)}|). \end{aligned}$$

Hence, $K_2 = 6$.

Consequently, from Theorem 3.1 it follows that $\mathcal{A} = \begin{pmatrix} \frac{6}{2^{R_1}} & \frac{6}{2^{R_2}} \\ \frac{6}{2^{R_1}} & \frac{6}{2^{R_2}} \end{pmatrix}$; and as the eigenvalues of the matrix $\mathcal{A}$ are 0 and $6(\frac{1}{2^{R_1}} + \frac{1}{2^{R_2}})$, the rates (R_1, R_2) that make the matrix $\mathcal{A}$ stable while the transmission rate $R = R_1 + R_2$ is minimum, are (R_1, R_2)= $(3, 5)$, $(5, 3)$, $(4, 4)$, in which for this system we choose $(R_1, R_2) = (4, 4)$. For these rates, $L_\infty^{(i)}$, $i = 1, 2$, are calculated as follows:

$$\begin{aligned} \sum_{j=0}^{t-1} \mathcal{A}^{t-1-j}((\mathcal{A}+\mathcal{B})\gamma + \omega) &= (\mathcal{A}+\mathcal{B})\gamma + \omega + \sum_{j=0}^{t-2} \mathcal{A}^{t-1-j}((\mathcal{A}+\mathcal{B})\gamma + \omega) \\ &= \begin{pmatrix} \frac{11}{8} \\ \frac{11}{8} \end{pmatrix} + \sum_{j=0}^{t-2} (\frac{3}{8})^{t-1-j} \begin{pmatrix} 2^{t-1-j-1} & 2^{t-1-j-1} \\ 2^{t-1-j-1} & 2^{t-1-j-1} \end{pmatrix} \begin{pmatrix} \frac{11}{8} \\ \frac{11}{8} \end{pmatrix} \\ &= \begin{pmatrix} E_t \\ E_t \end{pmatrix}, \end{aligned}$$

$$E_t \doteq \frac{11}{8} + \sum_{j=0}^{t-2} (\frac{3}{8})^{t-1-j} \frac{11}{4} 2^{t-1-j-1} = \frac{11}{8} + \frac{11}{8} 3(1 - (\frac{3}{4})^{t-1}),$$

and hence, $L_\infty^{(i)} = \lim_{t \to \infty} E_t = \frac{44}{8}$.

Fig. 3 illustrates the state trajectories of the system (8) and Fig. 4 illustrates the control trajectories when the proposed encoder, decoder and controller are used with rates $(R_1, R_2) = (4, 4)$ ($R = 8$ bits). As is clear from Fig. 3 by increasing time, the state trajectories of the system enter to the interval $[-\frac{44}{8}, \frac{44}{8}]$ and stay there despite of uncertainties in dynamic model and distortion due to quantization.

We can shrink the close bounded set $\Delta \subset \mathbb{R}^2$ by choosing larger rates. For example, for $(R_1, R_2) = (6, 6)$, $L_\infty^{(i)} = 1.6231$, $i = 1, 2$. Fig. 5 illustrates the state trajectories for this case.

Fig. 6 illustrates the state trajectories of the system when the proposed encoder, decoder and

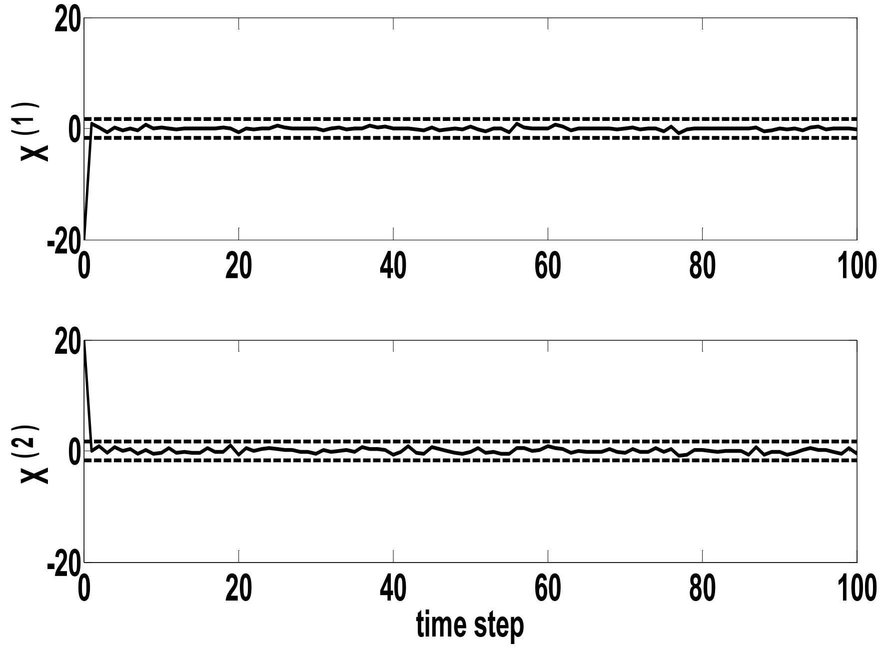

Fig. 5. The state trajectories for $(R_1, R_2)= (6, 6)$. Dashed lines identify the boundaries of the interval $[-1.6231, 1.6231]$.

controller with rates $(R_1, R_2)= (4, 3)$ ($R = 7$ bits) is used. As is clear from Fig. 6, for these rates, the proposed stabilizing technique is not able to stabilize the system. This result is expected as the rates $(R_1, R_2)= (4, 3)$ do not make the matrix $\mathcal{A}$ stable.

Fig. 7 illustrates the state trajectories of the system when $W_t^{(i)} = V_t^{(i)} = 0$, $i = \{1, 2\}$ and $(R_1, R_2) = (4, 4)$. As is clear from Fig. 7 by increasing time, the state trajectories, as expected from the Proposition 3.2, converge to zero.

V. Conclusion

This paper was concerned with the stability of nonlinear Lipschitz systems subject to bounded process and measurement noises when transmission from sensor to controller is subject to quantization distortion. A stabilizing technique and a sufficient condition relating transmission rate R to Lipschitz coefficients K_is were presented for almost sure asymptotic bounded stability of nonlinear uncertain Lipschitz systems. It was shown that in the absence of process and measurement noises, the proposed stabilizing technique results in almost sure asymptotic stability. Furthermore, it was illustrated via computer simulations that the proposed stabilizing technique has satisfactory performance for almost sure asymptotic bounded stability and asymptotic stability.

For future, it is interesting to also consider the effects of random packet dropout in transmission from sensor to controller on almost sure asymptotic bounded stability of nonlinear uncertain

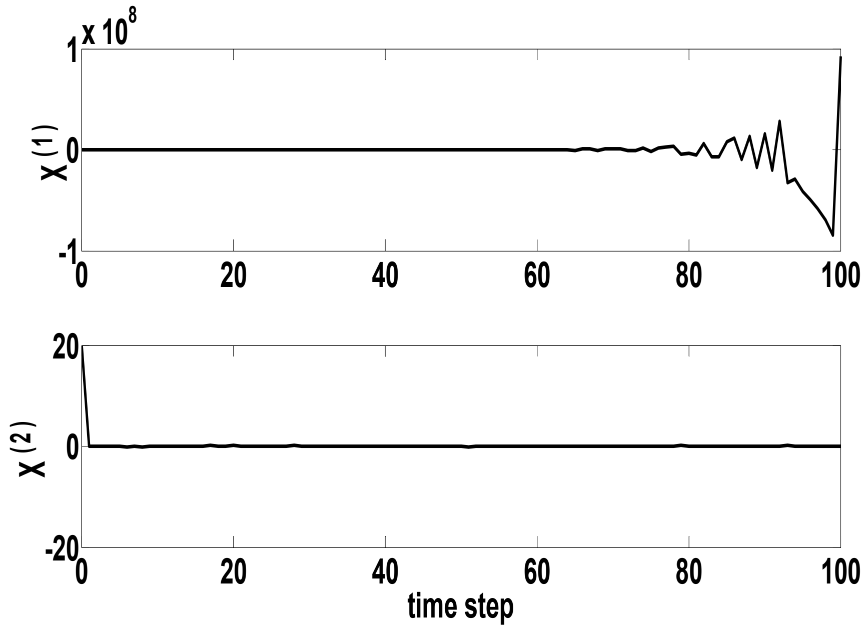

Fig. 6. The state trajectories for (R_1, R_2)= $(4, 3)$.

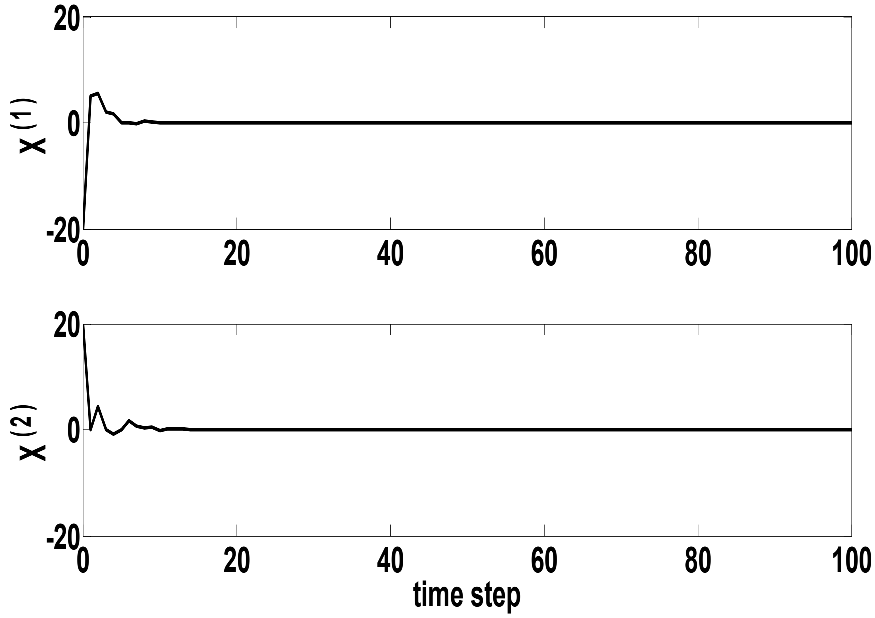

Fig. 7. The state trajectories for (R_1, R_2)= $(4, 4)$ and $W_t^{(i)} = V_t^{(i)} = 0$.

Lipschitz systems. It is also interesting to relax the assumption made on the matrix B.

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