

Robust Stabilizing Scheme for Uncertain Systems Controlled Over Limited Capacity Additive White Gaussian Noise Channels

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Abstract—In this paper we propose an encoding scheme and a stability scheme for reliable data reconstruction and robust stability of uncertain dynamical systems controlled over Additive White Gaussian Noise (AWGN) channels subject to limited capacity constraints. The uncertainty in the dynamical system is described by a relative entropy constraint. Specifically, the design of an encoder, decoder and controller that stabilize an uncertain system subject to the sum quadratic uncertainty description, is considered.

I. INTRODUCTION

In this paper, we are concerned with the control/communication system of Fig. 1. The control/communication system of Fig. 1 is defined on a complete probability space $(\Omega, \mathcal{F}(\Omega), P)$ with filtration $\{\mathcal{F}\}_{t \geq 0}$; $t \in \mathbb{N}_+ \triangleq \{0, 1, 2, \dots\}$, where Y_t , Z_t , \tilde{Z}_t , \tilde{Y}_t and U_t , $t \in \mathbb{N}_+$ are Random Variables (R.V.'s) denoting the source message (observations produced by sensors from the uncertain controlled dynamical system), channel input codeword, channel output codeword, the reproduction of the source message, and the control input to the source, respectively. The information source is the output of an uncertain controlled dynamical system with input U_t , output Y_t and state variables X_t . Throughout, it is assumed that the controlled dynamical system is subject to certain unknown terms (known as perturbed terms) which are unknown; but they belong to certain known classes. Different values for the perturbed terms correspond to different controlled dynamical systems. In this paper we are interested to those uncertain controlled dynamical systems which are described by a relative entropy constraint.

The control/communication system of Fig. 1 can correspond to tele-operation systems of space exploration devices. In such systems the communication from exploring device to controller (located at the base station) is subject to limited capacity constraint due to normally limited power supply of the exploring device; while the communication from controller to exploring device is unconstrained. Furthermore, in such applications, the exploring device is normally supposed to work in an unknown/uncertain environment. That is, the information source is also subject to uncertainty. The control/communication systems similar to the control/communication system of Fig. 1 have been

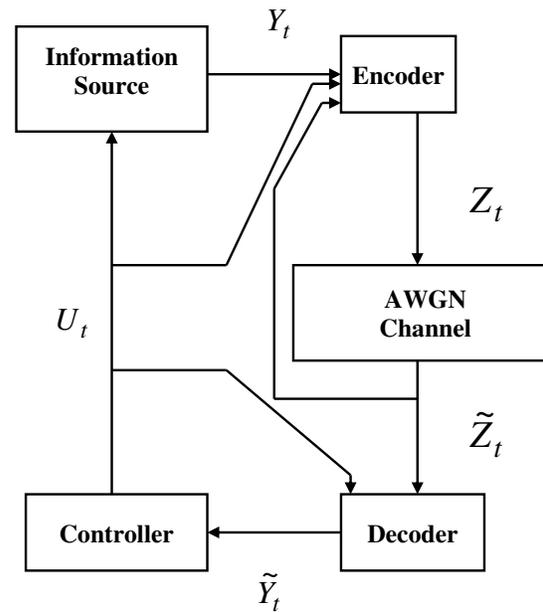


Fig. 1. Control/communication system subject to uncertainty in the source of information

considered in many places (e.g., [1]-[13]) to address reliable data reconstruction (known as observability in the literature) and stability questions of the dynamical systems which are controlled over limited capacity communication channels.

The objective of this paper is to design an encoder, decoder and controller which guarantee uniform mean square reconstruction of Y_t by \tilde{Y}_t and robust stability of the uncertain systems.

The problem of uniform observability and robust stability of fully observed uncertain dynamical systems subject to a uniformly bounded disturbance input is considered in [4], [9], [12], [13]. This paper complements the already existing results in the literature since it addresses similar questions for a class of dynamical systems which is described by a relative entropy constraint. This uncertainty description is a generalization of the sum quadratic uncertainty description considered in [14], [15], [16]. Sum quadratic uncertainty description includes uniformly bounded uncertainty description as a special case.

This paper is organized as follows. In Section II, the problem formulation is given. In Section III, a robust stabilizing scheme for stability of the fully observed uncertain dynamical systems is presented. Subsequently, in

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Section IV an encoding scheme is proposed that guarantee uniform reliable data reconstruction when the capacity is as minimum as possible. Finally, in Section V we conclude the paper.

II. PROBLEM FORMULATION

In this paper, we are concerned with the control/communication system of Fig. 1. Throughout, sequences of R.V.'s are denoted by $Y^T \triangleq (Y_0, Y_1, \dots, Y_T)$ for $T \in \mathbf{N}_+$ and $\log(\cdot)$ denotes the natural logarithm. A stochastic kernel $P(dF; x)$ is a mapping $P: \hat{\mathcal{A}} \times \bar{\mathcal{A}} \rightarrow [0, 1]$ which satisfies i) For every $x \in \bar{\mathcal{A}}$, the set function $P(\cdot; x)$ is a probability measure on $\hat{\mathcal{A}}$, and ii) For every $F \in \hat{\mathcal{A}}$, the function $P(dF; \cdot)$ is $\bar{\mathcal{A}}$ -measurable ($(\bar{\mathcal{A}}, \bar{\mathcal{A}})$, $(\hat{\mathcal{A}}, \hat{\mathcal{A}})$ are measurable spaces). $\sigma\{\cdot\}$ denotes σ -algebra, I_d denotes $(d \times d)$ identity matrix and $'$ denotes matrix transpose.

The different blocks of Fig. 1 are described below.

Information Source. The information source is the output of an uncertain controlled dynamical system with input U_t and output Y_t . Throughout, it is assumed that the controlled dynamical system is subject to perturbed terms which are unknown; but they belong to certain known classes. Let P denote the probability measure associated with the uncertain controlled dynamical system. In this paper we are interested to those uncertain systems which are described by the following relative entropy constraint.

$$P \in \mathcal{D}_{SU}(\Pi) \triangleq \left\{ P; \frac{1}{T} H(P||\Pi) \leq R_c + E_P \left[\frac{1}{2T} \sum_{t=0}^{T-1} H_t' M H_t \right] \right\} \quad (1)$$

where $H(\cdot||\cdot)$ is the relative entropy [17], $P(dX^{T-1} \times dY^{T-1})$ and $\Pi(dX^{T-1} \times dY^{T-1})$ are the probability measures associated with the uncertain system and the nominal system (i.e., the controlled dynamical system in the absence of the perturbed terms), respectively, R_c is a non-negative scalar, $H_t \in \mathbb{R}^d$ is the signal to be controlled, $M = M' \in \mathbb{R}^{d \times d}$ is positive semi-definite, and $E_P[\cdot]$ is the expectation with respect to the measure P .

The relative entropy $H(P||\Pi)$ can be thought of as a measure of the difference between the nominal probability measure Π and the perturbed probability measure P . Typical perturbation allowed under the above relative entropy constraint are the perturbations in the mean of the measure Π [16]. One example of such perturbations is given by the following class of Gauss Markov systems.

$$(\Omega, \mathcal{F}(\Omega), P; \{\mathcal{F}_t\}_{t \geq 0}) : \begin{cases} X_{t+1} = AX_t + NU_t + BW_t + B\bar{W}_t, & X_0 = X \\ Y_t = H_t, & H_t = X_t \end{cases} \quad (2)$$

where $X_t \in \mathbb{R}^d$, $U_t \in \mathbb{R}^o$, $W_t \in \mathbb{R}^m$, $\bar{W}_t \in \mathbb{R}^m$, $Y_t \in \mathbb{R}^d$, $H_t \in \mathbb{R}^d$, $X_0 \sim N(\bar{x}_0, \bar{V}_0)$, W_t is Independent Identically Distributed (i.i.d.) $\sim N(0, \Sigma_W)$, $\Sigma_W > 0$, and \bar{W}_t is the perturbed noise random process which is $\{\sigma\{W_l\}; l \leq t-1\}$ adapted.

The nominal system associated with the above uncertain system is the following system.

$$(\Omega, \mathcal{F}(\Omega), \Pi; \{\mathcal{F}_t\}_{t \geq 0}) : \begin{cases} X_{t+1} = AX_t + NU_t + BW_t, & X_0 = X \\ Y_t = H_t, & H_t = X_t \end{cases} \quad (3)$$

It can be shown that for the uncertain system (2) with the corresponding nominal system (3), $H(P||\Pi) = \frac{1}{2} E_P \left[\sum_{t=0}^{T-2} \bar{W}_t' \Sigma_W^{-1} \bar{W}_t \right]$ [16]. That is, the relative entropy constraint (1) holds for the uncertain system (2) with the nominal system (3), provided the following sum quadratic constraint holds.

$$\frac{1}{2T} E_P \left[\sum_{t=0}^{T-2} \bar{W}_t' \Sigma_W^{-1} \bar{W}_t \right] \leq R_c + E_P \left[\frac{1}{2T} \sum_{t=0}^{T-1} H_t' M H_t \right]. \quad (4)$$

Communication Channel: The communication channel is an AWGN channel with channel input Z_t , channel output \tilde{Z}_t and the channel noise \tilde{W}_t which is i.i.d. $\sim N(0, W_c)$. This communication channel at time t is subject to the power constraint $E_P[Z_t' Z_t] \leq P_t$, and it is described by $\tilde{Z}_t = Z_t + \tilde{W}_t$ where $Z_t, \tilde{Z}_t, \tilde{W}_t \in \mathbb{R}^d$.

Encoder: The encoder at any time $t \in \mathbf{N}_+$ is modeled by a stochastic kernel $P(dZ_t; y^t, u^{t-1}, \tilde{z}^{t-1})$.

Decoder: The decoder at any time $t \in \mathbf{N}_+$ is modeled by a stochastic kernel $P(d\tilde{Y}_t; \tilde{z}^t, u^{t-1})$.

Controller: The controller at any time $t \in \mathbf{N}_+$ is modeled by a stochastic kernel $P(dU_t; \tilde{z}^{t-1}, u^{t-1})$.

In this paper, we are concerned with the following observability and stability criteria.

Definition 2.1: (Uniform Mean Square Observability). Consider the control/communication system of Fig. 1 described by a class of dynamical systems. For a finite $D_v \geq 0$, the system is uniformly reconstructed using a mean square error criterion if there exist a control sequence, an encoder and decoder such that

$$\lim_{T \rightarrow \infty} \sup_{P \in \mathcal{D}_{SU}(\Pi)} \frac{1}{T} \sum_{t=0}^{T-1} E \|Y_t - \tilde{Y}_t\|^2 \leq D_v. \quad (5)$$

Definition 2.2: (Robust Stability). Consider the control/communication system of Fig. 1 described by a class of dynamical systems. Let $Y_t = H_t + \Gamma_t$ where H_t is the signal to be controlled and Γ_t is a function of the measurement noise and the perturbed terms. For a finite $D_v \geq 0$, the system is stabilizable if there exists an encoder, decoder and controller such that

$$\lim_{T \rightarrow \infty} \sup_{P \in \mathcal{D}_{SU}(\Pi)} \frac{1}{T} \sum_{t=0}^{T-1} E \|H_t\|^2 \leq D_v. \quad (6)$$

The objective of this paper is to design an encoding scheme and a stability scheme for uniform mean square observability and robust stability (as described above) when the capacity for uniform observability is as minimum as possible.

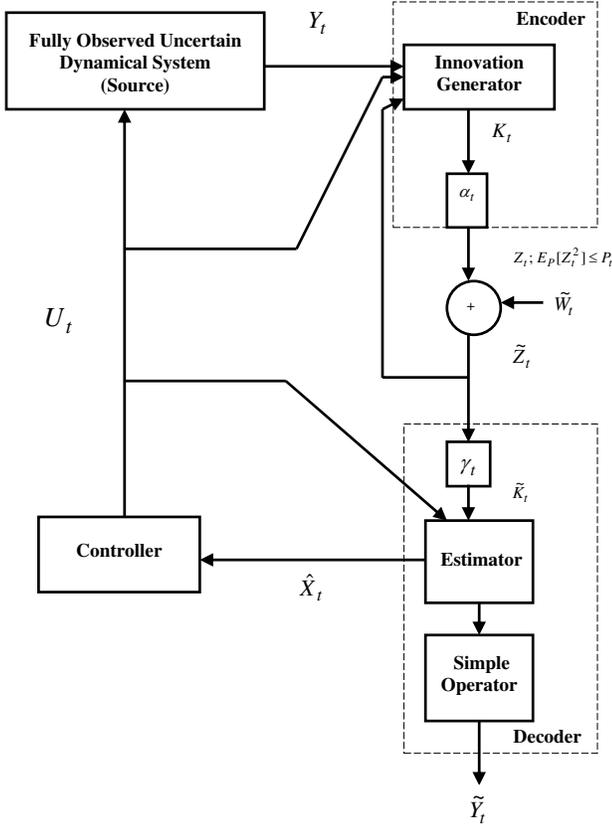


Fig. 2. Control/communication system subject to uncertainty in the source

III. ROBUST STABILITY OF AN UNCERTAIN FULLY OBSERVED SYSTEM

In this section we are concerned with the control/communication system of Fig. 2. In Fig. 2, the source is described by the fully observed uncertain systems described via the relative entropy constraint (1) with the following nominal system.

$$(\Omega, \mathcal{F}(\Omega), \Pi; \{\mathcal{F}_t\}_{t \geq 0}) : \begin{cases} X_{t+1} = AX_t + NU_t + BW_t, & X_0 = X, \\ Y_t = H_t, & H_t = X_t \end{cases} \quad (7)$$

where $X_t \in \mathbb{R}^d$, $U_t \in \mathbb{R}^o$, $W_t \in \mathbb{R}^m$, $Y_t \in \mathbb{R}^d$, $H_t \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$, $N \in \mathbb{R}^{d \times o}$, $B \in \mathbb{R}^{d \times m}$, $X_0 \sim N(\bar{x}_0, \bar{V}_0)$, W_t i.i.d. $\sim N(0, \Sigma_W)$ and $\Sigma_W > 0$. The encoder consists of a pre-encoding scheme that produces $K_t = Y_t - \hat{Y}_t$ where $\hat{Y}_t \triangleq \hat{X}_t \in \mathbb{R}^d$ is the mean square estimation of the observations process in the presence of uncertainty, in which this estimation is obtained by the knowledge of U^{t-1} and \tilde{Z}^{t-1} at the encoder (see Fig. 2). Then, the encoder scales K_t by $\alpha_t \in \mathbb{R}^{d \times d}$ and produces $Z_t = \alpha_t K_t$.

The decoder scales the channel outputs by $\gamma_t \in \mathbb{R}^{d \times d}$ and produces $\tilde{K}_t = \gamma_t \tilde{Z}_t$; and subsequently, it produces $\tilde{Y}_t = \tilde{K}_t + \hat{Y}_t$.

In this section we propose a stabilizing controller for robust

stability of the uncertain systems. In the next section, the parameters α_t and γ_t are chosen such that the process \tilde{K}_t is matched to the AWGN Channel (as described in [18]).

At the communication end, the process \tilde{K}_t can be viewed as the observations process of the partially observed uncertain dynamical systems described via the relative entropy constraint (1) with the following nominal system.

$$(\Omega, \mathcal{F}(\Omega), \Pi; \{\mathcal{F}_t\}_{t \geq 0}) : \begin{cases} X_{t+1} = AX_t + NU_t + BW_t, & X_0 = X, \\ \tilde{K}_t = \gamma_t \alpha_t H_t - \gamma_t \alpha_t \hat{Y}_t + \gamma_t \tilde{W}_t, \end{cases} \quad (8)$$

Next, consider the following pay-off functional

$$J = \lim_{T \rightarrow \infty} \frac{1}{2T} \sum_{t=0}^{T-1} E_P[X_t' X_t + U_t' R U_t], \quad (9)$$

where $R = R' \in \mathbb{R}^{o \times o} > 0$ and $U_t \in \mathcal{G}_{t-1}^U = \sigma\{\tilde{K}^{t-1}, U^{t-1}\}$. The objective is to find a control sequence that minimizes the maximum of the pay-off functional (9) over the class (1) (the Mini-max problem). Following similar methodology used in [16] by implementing the Legendre-Fenchel transformation [19], we can convert the Mini-max problem to an equivalent partial information, risk sensitive optimal control problem. Subsequently, the optimal controller is given by the followings.

$$U_t = -R^{-1} N' (\Pi_\infty^{-1} + N R^{-1} N' - \frac{B \Sigma_W B'}{\tau})^{-1} A (I_d - \frac{\Sigma_\infty \Pi_\infty}{\tau})^{-1} \hat{X}_t - \frac{\Sigma_\infty \Pi_\infty}{\tau} \hat{X}_t \quad (10)$$

$$\hat{X}_{t+1} = A \hat{X}_t + N U_t + K_\infty \tilde{K}_t + A (\Sigma_\infty^{-1} + \alpha_\infty' W_c^{-1} \alpha_\infty - \frac{I_d}{\tau} - M)^{-1} (\frac{I_d}{\tau} + M) \hat{X}_t, \quad \hat{X}_0 = \bar{x}_0,$$

$$K_\infty = A (\Sigma_\infty^{-1} + \alpha_\infty' W_c^{-1} \alpha_\infty - \frac{I_d}{\tau} - M)^{-1} \cdot \alpha_\infty' W_c^{-1} \alpha_\infty, \quad (11)$$

where Π_∞ and Σ_∞ are the solutions to the following indefinite Algebraic Riccati equations.

$$\Pi_\infty = A' \Pi_\infty A - A' \Pi_\infty \left(\Pi_\infty + \left(\frac{B \Sigma_\infty B'}{\tau} - N R^{-1} N' \right)^{-1} \Pi_\infty A + I_d + \tau M \right)^{-1} \Pi_\infty A + I_d + \tau M$$

$$\Sigma_\infty = A \Sigma_\infty A' - A \Sigma_\infty \left(\Sigma_\infty + \left(\frac{1}{\tau} I_d + M - \alpha_\infty' W_c^{-1} \alpha_\infty \right)^{-1} \Sigma_\infty A' + B \Sigma_W B' \right)^{-1} \Sigma_\infty A' + B \Sigma_W B'. \quad (12)$$

in which Π_∞ and Σ_∞ must satisfy the following conditions

$$\begin{aligned} \Sigma_\infty^{-1} + \alpha_\infty' W_c^{-1} \alpha_\infty - \frac{I_d}{\tau} - M &> 0, \\ \Sigma_\infty &> 0 \\ \Pi_\infty^{-1} - \frac{B \Sigma_W B'}{\tau} &> 0 \\ \Pi_\infty^{-1} - \frac{\Sigma_\infty}{\tau} &> 0 \end{aligned} \quad (13)$$

Furthermore, $\tau > 0$ is the lagrange multiplier which minimizes D^* given as follows

$$\begin{aligned} D^* &= 2\tau \left(\lim_{T \rightarrow \infty} \frac{\tilde{V}_\tau}{T} + R_c \right), \\ \lim_{T \rightarrow \infty} \frac{\tilde{V}_\tau}{T} &= -\frac{1}{2} \log(\det(I_d - (\frac{I_d}{\tau} + M)\Sigma_\infty)) \\ &\quad - \frac{1}{2} \log \det \Theta_\infty, \end{aligned} \quad (14)$$

where $\Theta_\infty = I_d - \frac{1}{\tau} K_\infty (\gamma_\infty W_c \gamma_\infty' + \gamma_\infty \alpha_\infty (\Sigma_\infty^{-1} - \frac{I_d}{\tau} - M)^{-1} \alpha_\infty' \gamma_\infty') K_\infty' (\Pi_\infty^{-1} - \frac{\Sigma_\infty}{\tau})^{-1}$.

Please note that D^* is the value of the cost functional (9) subject to the minimizing control sequence (10) that minimizes the maximum of the pay off functional (9) over the class (1).

Remark 3.1: i) The proposed stability scheme guarantees stability in the form of $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} E_P \|X_t\|^2 \leq D_v^c$ ($D_v^c \geq D^*$, when R is small) for all uncertain controlled dynamical systems which belong to the relative entropy constraint (1) with the nominal system (7).

ii) $R_c = 0$ and $M = 0$ (i.e., the case without uncertainty) correspond to the case where $\tau \rightarrow \infty$. For this case the results given in (10)-(12) are reduced to the standard Linear Quadratic Gaussian (LQG) results; and α_t and γ_t are given in [11].

iii) Since the uncertain system (2) subject to the sum quadratic constraint (4) is a sub-class of the relative entropy constraint (1) with the nominal system (7), then the proposed stability scheme also stabilizes such uncertain system.

IV. ENCODING SCHEME FOR UNIFORM OBSERVABILITY

In the previous section we proposed a stabilizing scheme that stabilizes the uncertain systems described via the relative entropy constraint (1) with the nominal system (7). In this section for the control/communication system of Fig. 2 described by such uncertain systems, we find α_t and γ_t such that the uniform mean square observability of Y_t by \hat{Y}_t up to the distortion value $D_v > 0$, is guaranteed when the capacity is as minimum as possible. For Simplicity in analysis we consider the case of $Y_t \in \mathfrak{R}$. The vector case is treated similarly. Please note that an example of uncertain systems described via the relative entropy constraint (1) with the nominal system (3), is the uncertain system (2) subject to the sum quadratic constraint (4).

Here, we use the source-channel matching technique [18]. That is, we find α_t and γ_t such that the stochastic kernel from K_t to \tilde{K}_t behaves like the minimizing kernel of the maximum rate distortion (i.e., $R_T^{sup}(D_v)$, see [20], (10) for definition).

The process $K_t \in \mathfrak{R}$ is an orthogonal process; and the process K_t associated with the maximum entropy (and subsequently maximum rate distortion) has the following distribution $K_t \sim N(0, \Sigma_t)$, where Σ_t is the uniform mean square estimation error in estimation of X_t by $\hat{Y}_t = \hat{X}_t$ when the system is subject to uncertainty. Σ_t is the solution of the

following indefinite Riccati equation.

$$\begin{aligned} \Sigma_{t+1} &= A \Sigma_t A - A \Sigma_t \left(\Sigma_t + \left(\frac{1}{\tau} + M - \frac{\alpha_\infty^2}{W_c} \right)^{-1} \right)^{-1} \\ &\quad \cdot \Sigma_t A, \\ \Sigma_0 &= \bar{V}_0. \end{aligned} \quad (15)$$

Subsequently, minimizing kernel associated with maximum rate distortion is given by

$$\begin{aligned} P^*(d\tilde{K}^{T-1}; k^{T-1}) &= \left(\prod_{t=0}^{T-1} q_{\tilde{K}_t|K_t}^* \right) d\tilde{K}^{T-1}, \\ q_{\tilde{K}_t|K_t}^* &\sim N(\eta_t k_t, \eta_t D_v), \eta_t = 1 - \frac{D_v}{\Sigma_t}, \end{aligned} \quad (16)$$

where $D_v < \min_{t \in \mathbf{N}_+} \Sigma_t$.

Following the solution (16), α_t and γ_t must be chosen as follows

$$\alpha_t = \sqrt{\frac{\eta_t W_c}{D_v}}, \quad \gamma_t = \sqrt{\frac{D_v \eta_t}{W_c}}, \quad \eta_t = 1 - \frac{D_v}{\Sigma_t}. \quad (17)$$

Subsequently, it is easily shown that $E_P(K_t - \tilde{K}_t)^2 \leq D_v$ over all values for the perturbed terms (and subsequently $E_P(Y_t - \tilde{Y}_t)^2 \leq D_v$). Furthermore, under assumption that the indefinite Riccati equation (15) converges to the associated indefinite Riccati equation (12) (conditions under which this convergence is guaranteed have been discussed in [21], Chapter 14), then $\mathcal{C} = \mathcal{H}_r(\mathcal{K}) - \frac{1}{2} \log_e(2\pi e D_v)$, where \mathcal{C} is the capacity of the AWGN channel and $\mathcal{H}_r(\mathcal{K})$ is the robust entropy rate of the process K_t (see [10], Definition 2.1).

Remark 4.1: Following the necessary condition presented in ([20], Theorem 2.4) $\mathcal{C} = \mathcal{H}_r(\mathcal{K}) - \frac{1}{2} \log(2\pi e D_v)$ is the minimum capacity for uniform mean square observability of the process K_t by \tilde{K}_t , up to the distortion value D_v .

Next, consider the control/communication system of Fig. 2 described by the uncertain system (2) subject to the sum quadratic constraint (4). The observations process of system (2) is written as follows.

$$\begin{aligned} Y_t &= A^t X_0 + \sum_{i=0}^{t-1} A^{t-1-i} B W_i + \sum_{i=0}^{t-1} A^{t-1-i} B \bar{W}_i \\ &\quad + \sum_{i=0}^{t-1} A^{t-1-i} N U_i, \quad 0 \leq t \leq T-1, \end{aligned} \quad (18)$$

where \bar{W}_t belongs to the following class

$$\begin{aligned} &\left\{ \{ \bar{W}_t \}_{t=0}^{T-2}; E_P \left[\frac{1}{2T} \sum_{t=0}^{T-2} (\bar{W}_t' \Sigma_W^{-1} \bar{W}_t) \right] \leq R_c \right. \\ &\quad \left. + E_P \left[\frac{1}{2T} \sum_{t=0}^{T-1} (Y_t' M Y_t) \right] \right\}. \end{aligned} \quad (19)$$

But, by the knowledge of the control sequence U^{t-1} at the decoder, the last term in (18) (i.e., $\sum_{i=0}^{t-1} A^{t-1-i} N U_i$) is reconstructed perfectly at the communication end. Thus, since U^{t-1} is also available in the encoder, the problem of reliable data reconstruction of Y_t is reduced to the equivalent

problem of reliable data reconstruction of \bar{Y}_t given by the following.

$$\bar{Y}_t = A^t X_0 + \sum_{i=0}^{t-1} A^{t-1-i} B W_i + \sum_{i=0}^{t-1} A^{t-1-i} B \bar{W}_i. \quad (20)$$

But, \bar{Y}_t is the outputs of the uncontrolled analogous system, that is, the following system.

$$\begin{aligned} & (\Omega, \mathcal{F}(\Omega), P; \{\mathcal{F}_t\}_{t \geq 0}) : \\ & \begin{cases} X_{t+1} = A X_t + B W_t + B \bar{W}_t, & X_0 = X, \\ \bar{Y}_t = H_t, & H_t = X_t \end{cases} \end{aligned} \quad (21)$$

where for this system, the perturbed term \bar{W}_t belongs to the following class.

$$\begin{aligned} & \left\{ \{\bar{W}_t\}_{t=0}^{T-2}; E_P \left[\frac{1}{2T} \sum_{t=0}^{T-2} (\bar{W}_t' \Sigma_W^{-1} \bar{W}_t) \right] \leq R_c \right. \\ & \left. + E_P \left[\frac{1}{2T} \sum_{t=0}^{T-1} (\bar{Y}_t' M \bar{Y}_t) \right] \right\}. \end{aligned} \quad (22)$$

But, this class is independent of the control sequence. Thus, without loss of generality in addressing the uniform observability of the control/communication system of Fig. 2 described by the uncertain system (2) subject to the sum quadratic constraint (4), we can consider the uncontrolled uncertain system (21) subject to the sum quadratic constraint (22).

In ([20], Theorem 4.1) the robust entropy rate of the observations process of the uncontrolled system (21) subject to the sum quadratic constraint (22) has been found, in which we summarize this result in the following theorem.

Theorem 4.2: Consider the uncertain system (21) subject to the sum quadratic constraint (22). Let for some $s \geq 0$, i) $B'(B \Sigma_W B')^{-1} B - (1+s) \Sigma_W^{-1} < 0$, ii) (A, B) is controllable, iii) A and $B'(B \Sigma_W B')^{-1} B - (1+s) \Sigma_W^{-1}$ are invertible, and iv) $\beta(\eta) > 0$ for some η such that $|\eta| = 1$ where $\beta(\eta)$ is the rational matrix function given by

$$\begin{aligned} \beta(\eta) &= B'(B \Sigma_W B')^{-1} B - (1+s) \Sigma_W^{-1} + B'(\eta^{-1} I_d \\ &\quad - A') s M (\eta I_d - A)^{-1} B, \quad s \geq 0. \end{aligned} \quad (23)$$

Then

$$\begin{aligned} \mathcal{H}_r(\bar{\mathcal{Y}}) &= \frac{d}{2} \log(2\pi e) + \frac{1}{2} \log \det(B \Sigma_W B') \\ &\quad + \min_{s \geq 0} \left\{ s R_c + \frac{1}{2} \text{trac}(B' \Xi_\infty B \Sigma_W) \right\} \end{aligned} \quad (24)$$

where Ξ_∞ is the solution of the following Algebraic Riccati equation appearing in the H^∞ estimation and control problem

$$\begin{aligned} \Xi_\infty &= A' \Xi_\infty A - A' \Xi_\infty B [B'(B \Sigma_W B')^{-1} B \\ &\quad - (1+s) \Sigma_W^{-1} + B' \Xi_\infty B]^{-1} B' \Xi_\infty A + s M. \end{aligned} \quad (25)$$

Please note that since the Algebraic Riccati equation (25) is quadratic, it has two symmetric solutions. Nevertheless, in (24) the solution under which $\{s R_c + \frac{1}{2} \text{trac}(B' \Xi_\infty B \Sigma_W)\}$ is bigger, must be used. Please note that conditions ii-iv

guarantee that the Algebraic Riccati equation (25) has real solutions. These conditions just need to be valid for $s \geq 0$ which minimizes $\{s R_c + \frac{1}{2} \text{trac}(B' \Xi_\infty B \Sigma_W)\}$. Furthermore, condition i is critical for the validity of the result. Similarly, this condition just need to be valid for $s \geq 0$ which minimizes $\{s R_c + \frac{1}{2} \text{trac}(B' \Xi_\infty B \Sigma_W)\}$. Please note that Condition i is invalidated for $M = 0$.

Corollary 4.3: The robust entropy rate calculated in Theorem 4.2 is very useful to determine if the proposed encoding/decoding scheme is optimal. For a given distortion value D_v if $\mathcal{H}_r(\mathcal{K}) \approx \mathcal{H}_r(\bar{\mathcal{Y}})$, then from the necessary condition presented in ([20], Theorem 2.4) applied to the process \bar{Y}_t ; and from the proposed encoding scheme follows that the capacity $\mathcal{C} = \mathcal{H}_r(\mathcal{K}) - \frac{1}{2} \log(2\pi e D_v)$ is also minimum for uniform observability of the observations process of the uncertain system (2) subject to the sum quadratic uncertainty constraint (4).

As we discussed, the condition i of Theorem 4.2 is invalidated for $M = 0$. Subsequently, this theorem can not present the robust entropy rate when $M = 0$. Subsequently, in the following remark we calculate the robust entropy rate when $M = 0$.

Remark 4.4: [22] The robust entropy rate of the uncertain uncontrolled systems described via the relative entropy constraint (1) with $M = 0$ and the uncontrolled version of the nominal system (3) (i.e., (3) with $U_t = 0$) is given as follows.

$$\mathcal{H}_r(\bar{\mathcal{Y}}) = \mathcal{H}_S(\bar{\mathcal{Y}}) + \frac{d}{2} \log\left(\frac{1+s^*}{s^*}\right), \quad (26)$$

where $\mathcal{H}_S(\bar{\mathcal{Y}}) = \frac{1}{2} \log((2\pi e)^d \det(B \Sigma_W B'))$ is the Shannon entropy rate of the nominal system; and for a given $R_c \in [0, \infty)$, $s^* > 0$ is the unique solution of the following nonlinear equation.

$$R_c = -\frac{d}{2} \log\left(\frac{1+s^*}{s^*}\right) + \frac{d}{2s^*}. \quad (27)$$

It is evident that the robust entropy rate (26) also represents the robust entropy rate of the system (21) subject to the sum quadratic constraint (22) when $M = 0$. This robust entropy rate is also very useful to determine if the proposed encoding/decoding scheme is optimal. Following the same discussion we had in Corollary 4.3 if $\mathcal{H}_r(\mathcal{K}) \approx \mathcal{H}_r(\bar{\mathcal{Y}})$, then the capacity $\mathcal{C} = \mathcal{H}_r(\mathcal{K}) - \frac{1}{2} \log(2\pi e D_v)$ is also minimum for the uniform observability of the observations process of the uncertain system (2) subject to the sum quadratic constraint (4) when $M = 0$.

V. CONCLUSION

In this paper for the control/communication system of Fig. 2 described by the uncertain systems characterized via the relative entropy constraint (1) with the nominal system (3), a stability scheme and an encoding scheme were proposed for robust stability in the form of $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} E_P \|X_t\|^2 \leq D_v^c$ for all possible values of the perturbed terms. Furthermore, the proposed encoding scheme guaranteed uniform mean square observability up to the distortion value $D_v > 0$ when the capacity was as minimum as possible. In the real

life applications the desired stability criterion (i.e., D_v^c) is given. This criterion determines the admissible distortion value (i.e., $D_v > 0$) from relation $D_v^c = D^*$ (where D^* is given by (14) when R is small). Subsequently, by implementing the proposed encoding scheme we have uniform observability up to the admissible distortion value D_v , while the capacity is as minimum as possible; and the robust stability is being guaranteed using the proposed controller. For the future direction it would be interesting to extend above results to the uncertain partially observed systems described by the relative entropy constraint (1). This extension can be done following the same methodology used in Section IV and by finding the robust entropy rate.

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