An Introduction to HILBERT-HUANG TRANSFORM and EMPIRICAL MODE DECOMPOSITION

(HHT-EMD)

Advanced Structural Dynamics
(CE 20162)

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Contents

Scope and Goals

- Need a basis that determined through signal characteristics , not by user prior to the analysis.

- Need transformation that be able to overcome Heisenberg uncertainty theorem which is restricted resolution.

- Need to noise subtraction.

Review on transformations

- TRADITIONAL FOURIER
- STFT:
  - which allow a signal to be nonstationary as long as it is piece-wise stationary
- WAVELET
  - which can sift out particular signatures from a signal on a variety of size scales
- HHT

STFT

- General STFT

$$STFT(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)h(t-\tau)e^{-i\omega\tau}d\tau$$

- Gabor Transform

$$G_x(t,f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau$$

Gaussian window function minimizes the Fourier uncertainty principle. (Gabor transform with modifications for multi-resolution becomes the Morlet wavelet transform).

- Generalized Time-Frequency Distributions

$$C(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\gamma,\tau)\, x\left(t+\frac{\tau}{2}\right) x^*(t-\frac{\tau}{2})e^{-i\omega\tau-i\gamma(\theta+t)}d\theta d\tau d\gamma$$

Overall, the Wigner-Ville distribution gives better time and frequency resolution than STFT and does not have to sacrifice one resolution for the benefit of the other.

WAVELET

$$F_{WT}(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right) dt$$

- This was an improvement over STFT because, as the result of the flexible basis functions, both the high-frequency and the low-frequency structures could be analyzed.
- However, a drawback of wavelet analysis is that the wavelet basis functions, and therefore the structures being sifted out from the original signal, are chosen *a priori*.
- It is possible that the utilized wavelets may or may not reflect the processes in the analyzed signal.

Introduction

- HHT is able to extract the frequency components from possibly nonlinear and nonstationary intermittent signals

- HHT is able to overcome Heisenberg uncertainty theorem which is restricted resolution of Fourier transformation.

- HHT has been used to study a wide variety of data including rainfall, earthquakes, heart-rate variability, financial time series, Lidar data, and ocean waves to name a few subjects.

HHT

Hilbert transform is developed the unique and physical definitions of instantaneous frequency and instantaneous amplitude of a signal but with different physical explanation of frequency , generalized from the conventional Fourier definition.

$$Y(t) = H[x(t)] = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x(\tau)}{t-\tau} d\tau$$

$x(t)$ arbitrary real signal
$Y(t)$ Hilbert transform of $x(t)$
$Z(t)$ analytic signal of $x(t)$ in polar coordinate representation

$$Z(t) = x(t) + iY(t) = a(t)e^{i\theta(t)}$$
$$a(t) = [x^2(t) + Y^2(t)]^{1/2}$$
$$\theta(t) = arctan\,\frac{Y(t)}{x(t)}$$
$$\omega(t) = \theta'(t)$$

$\omega(t)$ is dominated as instantaneous frequency
$a(t)$ the instantaneous amplitude

Mathematical Properties of Hilbert

- Linearity

$$Hf(t) = c_1 Hf_1(t) + c_2 Hf_2(t).$$

- Inverse of Hilbert Transform

$$H^2 = I$$
$$H^3 = H^{-1}$$

- Derivatives of Hilbert Transform

$$H\big(f'(t)\big) = H'\big(f(t)\big)$$

- Orthogonality

$$\int_{-\infty}^{\infty} f(t)\widehat{f}(t)dt = 0,$$

A real function $f(t)$ and its Hilbert transform $\widehat{f}(t)$ are orthogonal

some Hilbert transform Pairs

$g(t)$	$\hat{g}(t)$
$a_1 g_1(t) + a_2 g_2(t);\ a_1, a_2 \in \mathbb{C}$	$a_1 \hat{g}_1(t) + a_2 \hat{g}_2(t)$
$h(t - t_0)$	$\hat{h}(t - t_0)$
$h(at);\ a \neq 0$	$\mathsf{sgn}(a)\hat{h}(at)$
$\frac{d}{dt}h(t)$	$\frac{d}{dt}\hat{h}(t)$
$\delta(t)$	$\frac{1}{\pi t}$
$e^{\mathrm{j}t}$	$-\mathrm{j}e^{\mathrm{j}t}$
$e^{-\mathrm{j}t}$	$\mathrm{j}e^{-\mathrm{j}t}$
$\cos(t)$	$\sin(t)$
$\mathrm{rect}(t)$	$\frac{1}{\pi}\ln\lvert(2t+1)/(2t-1)\rvert$
$\mathrm{sinc}(t)$	$\frac{\pi t}{2}\mathrm{sinc}^2(t/2) = \sin(\pi t/2)\,\mathrm{sinc}(t/2)$
$1/(1+t^2)$	$t/(1+t^2)$

Comparison b/w transformers (.wiki)

Transform	Fourier	Wavelet	Hilbert
Basis	a priori	a priori	adaptive
Frequency	convolution: global, uncertainty	convolution: regional, uncertainty	differentiation: local, certainty
Presentation	energy-frequency	energy-time-frequency	energy-time-frequency
Nonlinear	no	no	yes
Non-stationary	no	yes	yes
Feature Extraction	no	discrete: no, continuous: yes	yes
Theoretical Base	theory complete	theory complete	empirical

HHT

- not all functions give “good” Hilbert transforms, meaning those which produce physical instantaneous frequencies.

- It is essentially an algorithm which decomposes nearly any signal into a finite set of functions which have “good” Hilbert transforms that produce physically meaningful instantaneous frequencies.

- For this purposes, the empirical mode decomposition was introduced.

IMF

1- the number of local extrema and the number of zero crossings must either equal or differ at most by one;

2- at any point, the mean value of the envelope defined by local maxima and the envelope defined by local minima is zero.

Any signal satisfying these two conditions is called an intrinsic mode function (IMF)

EMD

EMD is which process is decomposed an arbitrary real signal into its IMFs by sifting.

1. Determine the local extrema (maxima, minima) of the signal.
2. Connect the maxima with an interpolation function, creating an upper
envelope about the signal.
3. Connect the minima with an interpolation function, creating a lower
envelope about the signal.
4. Calculate the local mean as half the difference between the upper and lower
envelopes.
5. Subtract the local mean from the signal.
6. Iterate on the residual.

EXAMPLE

by patrick Flandrin

Limitations for IF computed through Hilbert Transform

- Data must be expressed in terms of Intrinsic Mode Function. IMF is only necessary but not sufficient.
- Bedrosian Theorem: Hilbert transform of $a(t) \cos \theta(t)$ might not be exactly $a(t) \sin \theta(t)$. Spectra of $a(t)$ and $\cos \theta(t)$ must be disjoint.
- Nuttall Theorem: Hilbert transform of $\cos \theta(t)$ might not be $\sin \theta(t)$ for an arbitrary function of $\theta(t)$. Quadrature and Hilbert Transform of arbitrary real functions are not necessarily identical.
- **Therefore, a simple derivative of the phase of the analytic function for an arbitrary function may not work.**

Bedrosian Theorem

Let f(x) and g(x) denotes generally complex functions in $L^2(-\infty, \infty)$ of the real variable x. If

(1) the Fourier transform $F(\omega)$ of f(x) vanished for $|\omega| > a$ and the Fourier transform $G(\omega)$ of g(x) vanishes for $|\omega| < a$, where a is an arbitrary positive constant, or

(2) f(x) and g(x) are analytic (i. e., their real and imaginary parts are Hilbert pairs),

then the Hilbert transform of the product of f(x) and g(x) is given

$$H\{ f(x)\, g(x) \} = f(x)\, H\{ g(x) \} .$$

Bedrosian, E., 1963: A Product theorem for Hilbert Transform, Proceedings of the IEEE, 51, 868-869.

Nuttall Theorem

For any function ***x(t),*** having a quadrature ***xq(t),*** and a Hilbert transform ***xh(t);*** then,

$$E = \int_0^{\infty} \left| xq(t) - xh(t) \right|^2 dt$$

$$= 2\int_{-\infty}^{0} \left| F_q(\omega) \right|^2 d\omega \, ,$$

where ***Fq(ω)*** is the spectrum of ***xq(t).***

Nuttall, A. H., 1966: On the quadrature approximation to the Hilbert Transform of modulated signal, Proc. IEEE, 54, 1458

Difficulties with the Existing Limitations

- Data are not necessarily IMFs.
- Even if we use EMD to decompose the data into IMFs. IMF is only necessary but not sufficient because of the following limitations:
- Bedrosian Theorem adds the requirement of not having strong amplitude modulations.
- Nuttall Theorem further points out the difference between analytic function and quadrature.
- The discrepancy, however, is given in term of the quadrature spectrum, which is an unknown quantity. Therefore, it cannot be evaluated. Nuttall Theorem provides a constant limit not a function of time; therefore, it is not very useful for non-stationary processes.

Analytic vs. Quadrature

For ***cosθ(t)*** with arbitrary function of ***θ(t) :***

$$X(t) \xrightarrow{\text{Hilbert Transform}} Y(t) \longrightarrow Z(t) \quad \textit{Analytic}$$

$$X(t) \xrightarrow{\text{No Known general method}} Q(t) \quad \textit{Quadrature, not analytic}$$

Analytic functions satisfy Cauchy-Reimann equation, but may be $x^2 + y^2 \neq 1$. Then the arc-tangent would not recover the true phase function.

Quadrature pairs are not analytic, but satisfy strict 90º phase shift; therefore, $x^2 + y^2 = 1$, and the arc-tangent always gives the true phase function.

Other Limitations

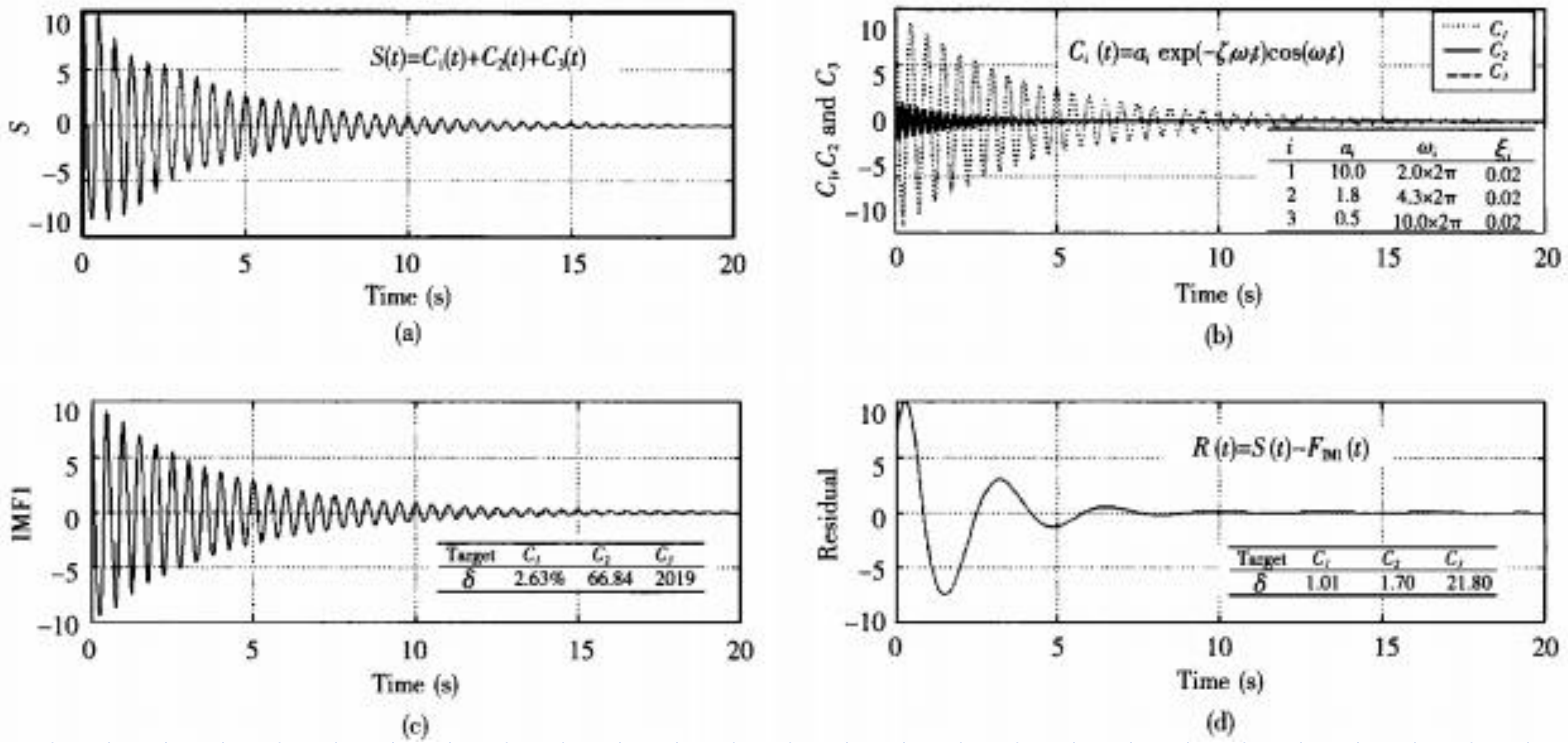
The EMD has limitations in distinguishing components in narrowband signals.

1- End effect issue,

2- Order of the IMF extractions,

3- EMD’s unsatisfactory resolution.

example



i	a_i	ω_i	ξ_i
1	10.0	$2.0\times2\pi$	0.02
2	1.8	$4.3\times2\pi$	0.02
3	0.5	$10.0\times2\pi$	0.02

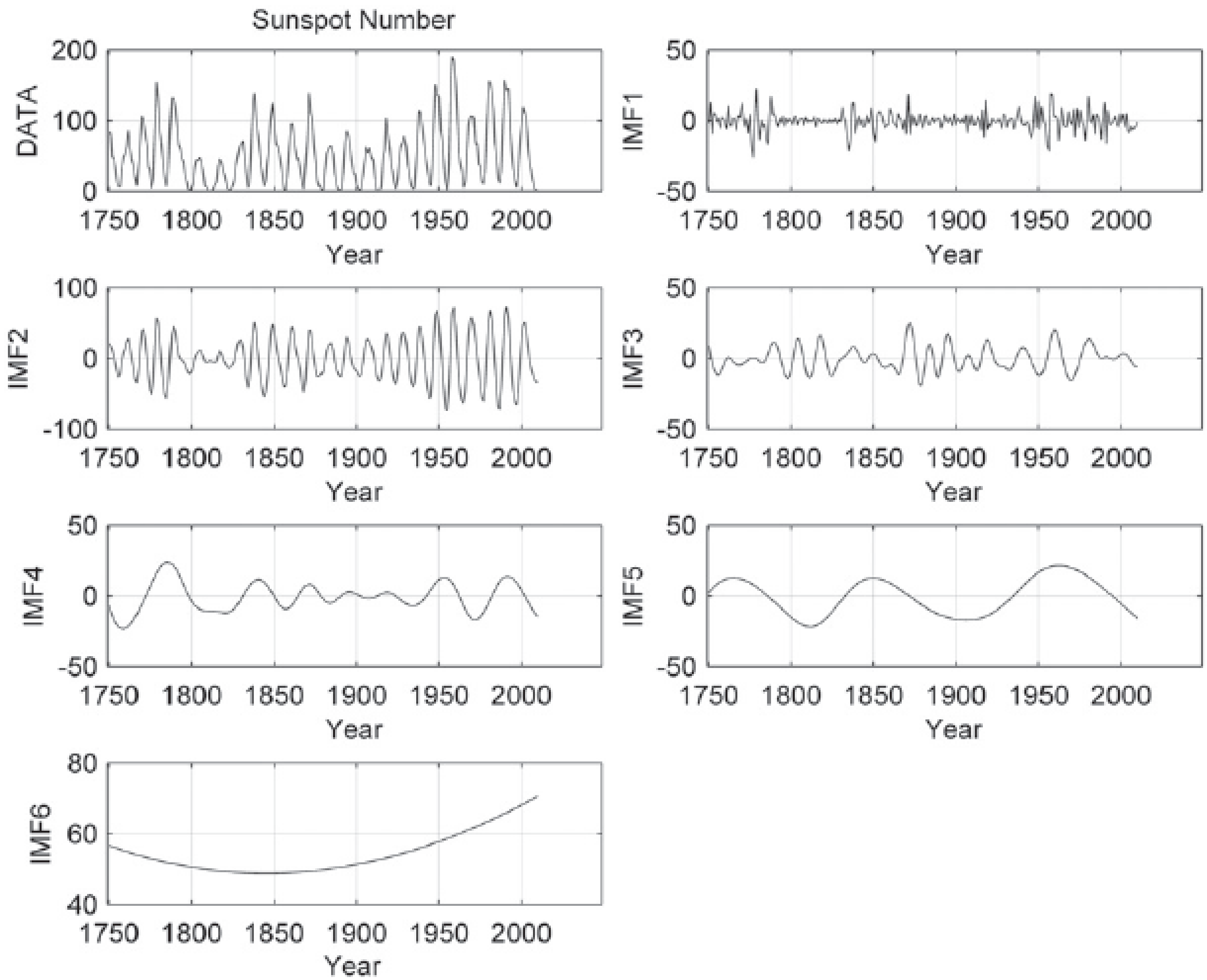
Target	C_1	C_2	C_3
δ	2.63%	66.84	2019

Target	C_1	C_2	C_3
δ	1.01	1.70	21.80

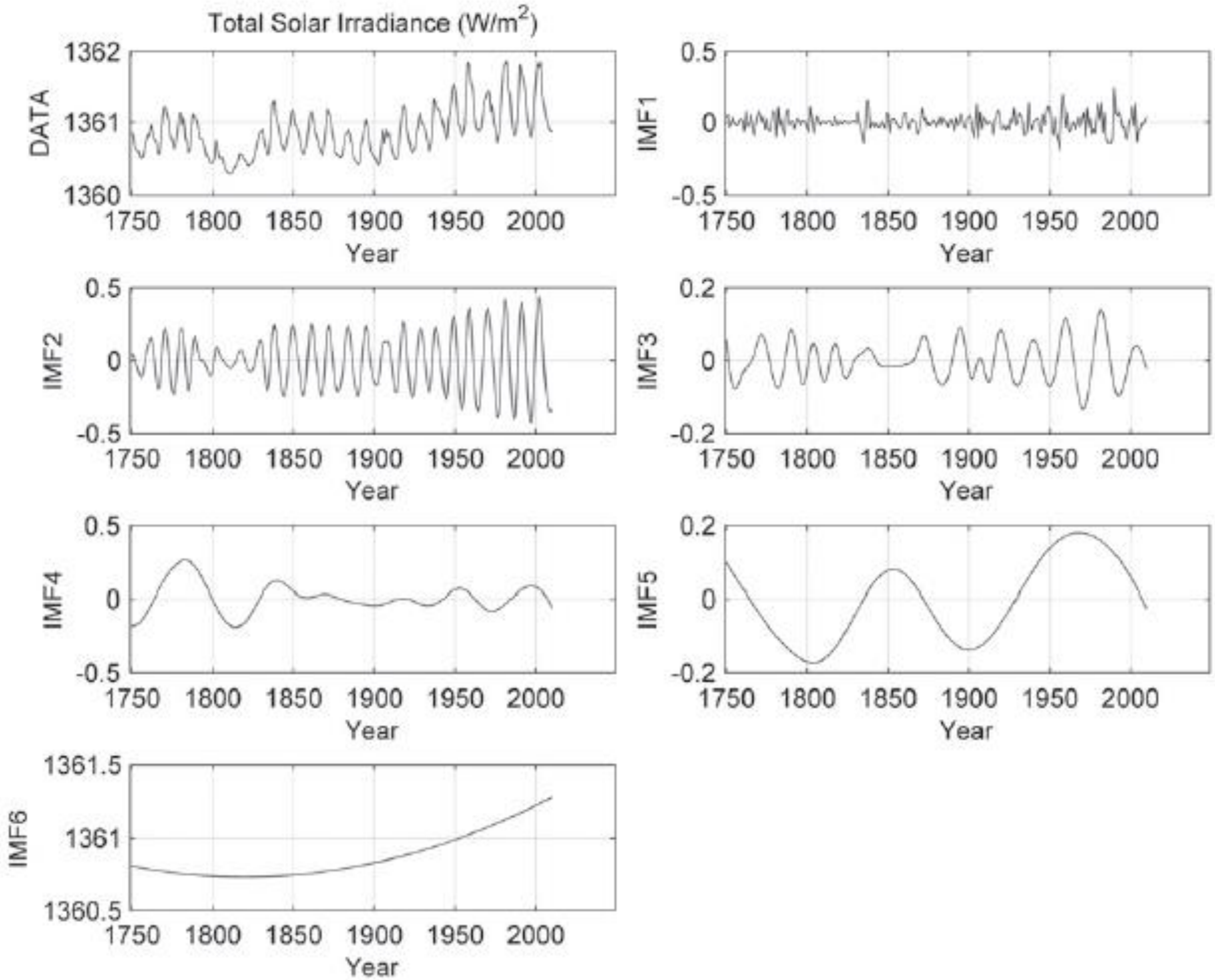
[Chen, 2003]proposes a new method to eliminate this problem

Example

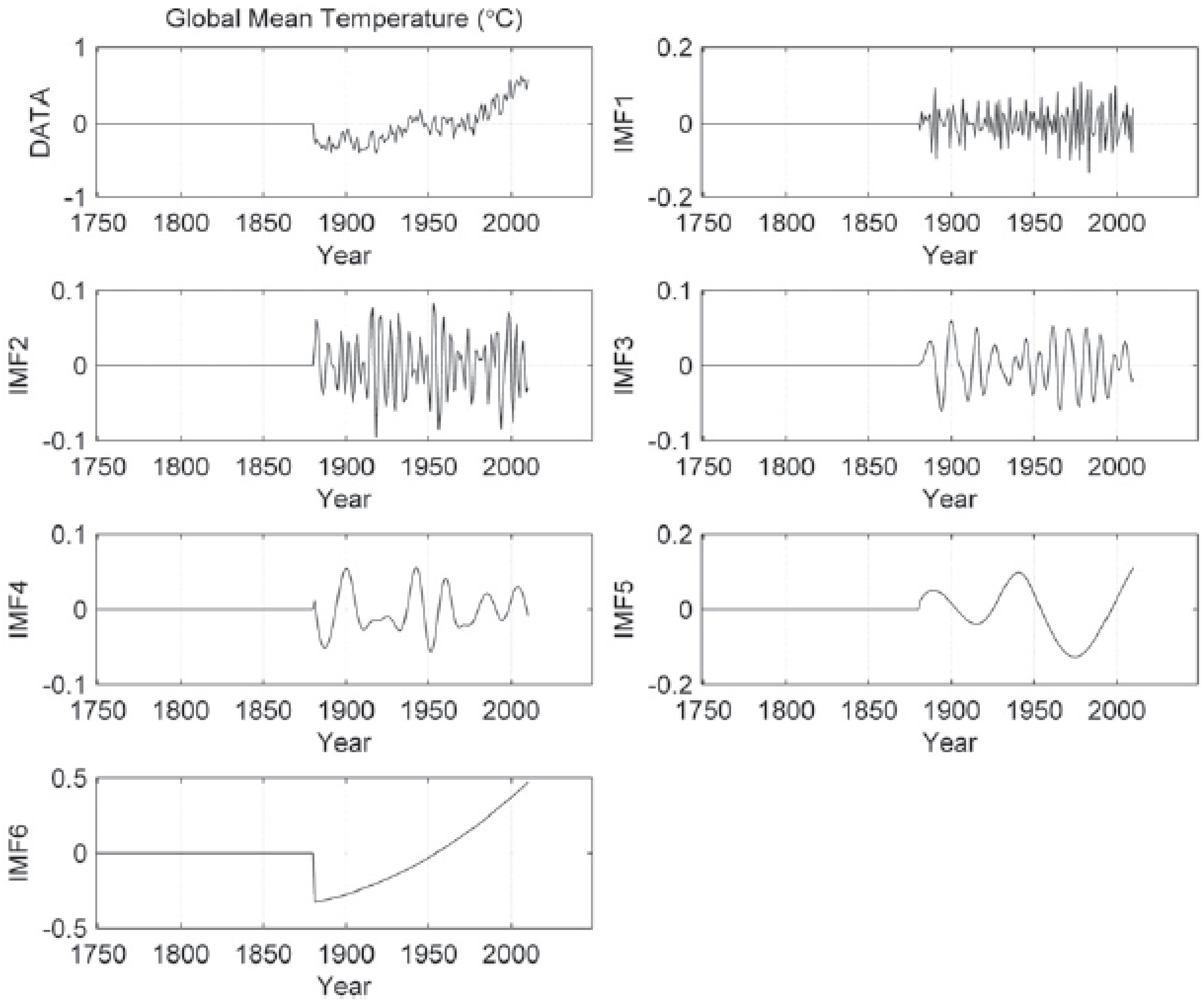
Sunspot number data set



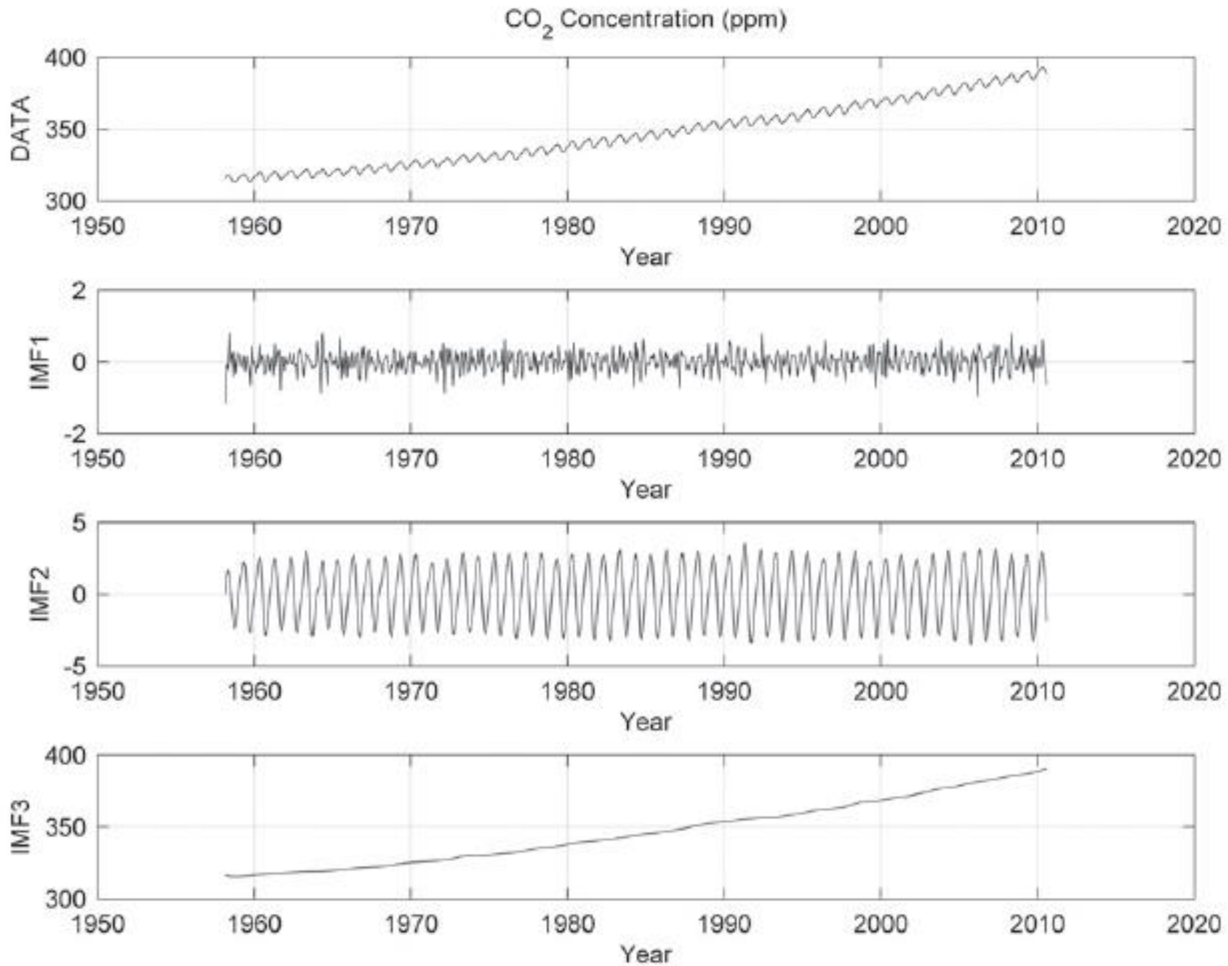
Total solar Irradiance measurements



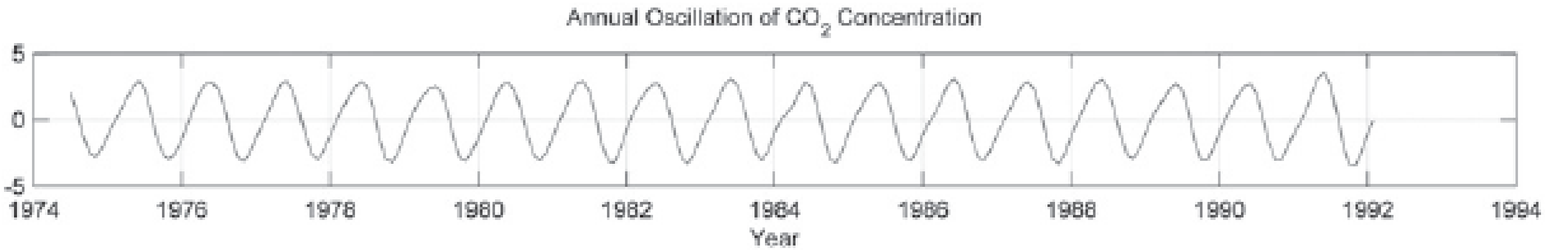
Global mean temperature



CO2 Concentration

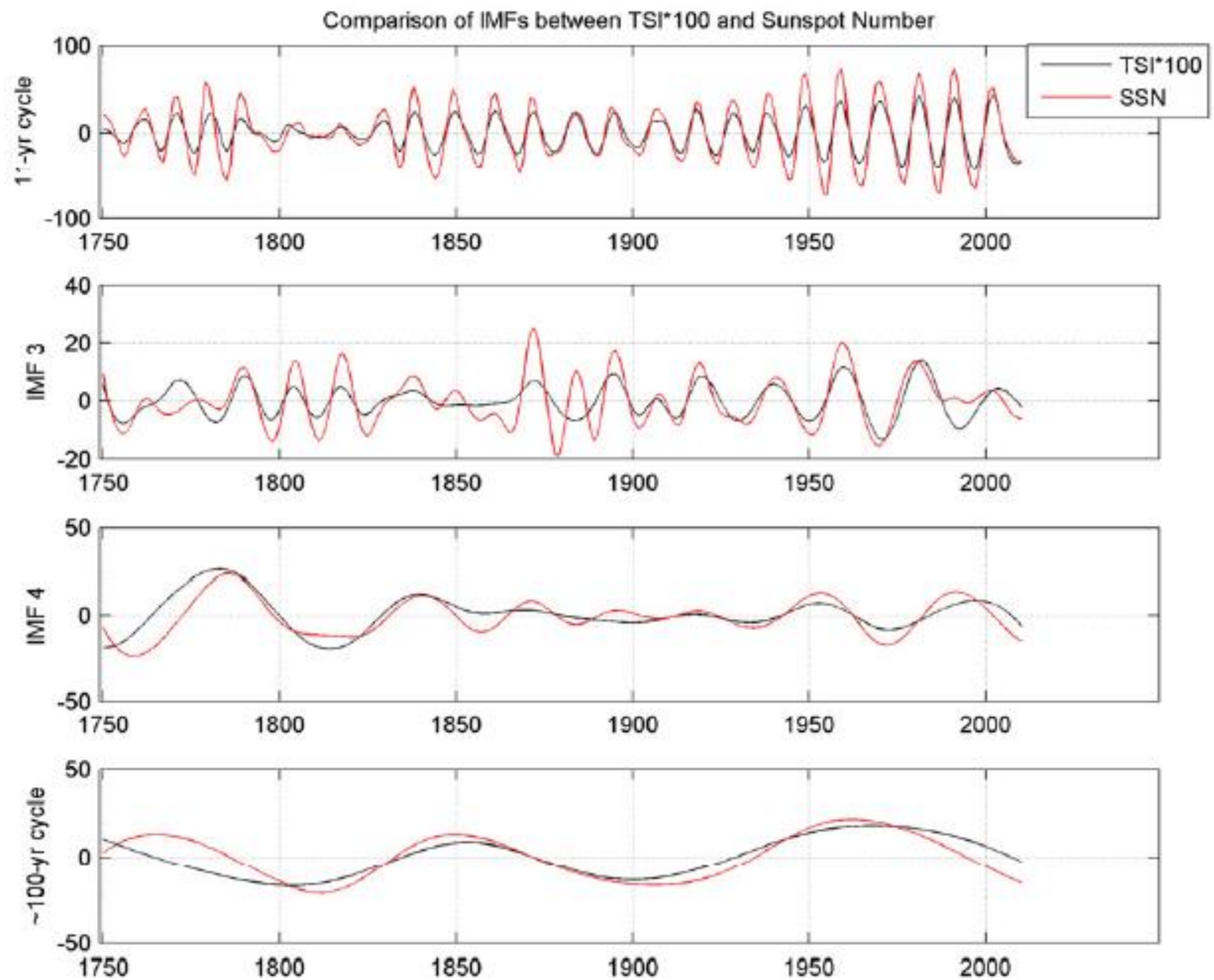


Subsection of IM2

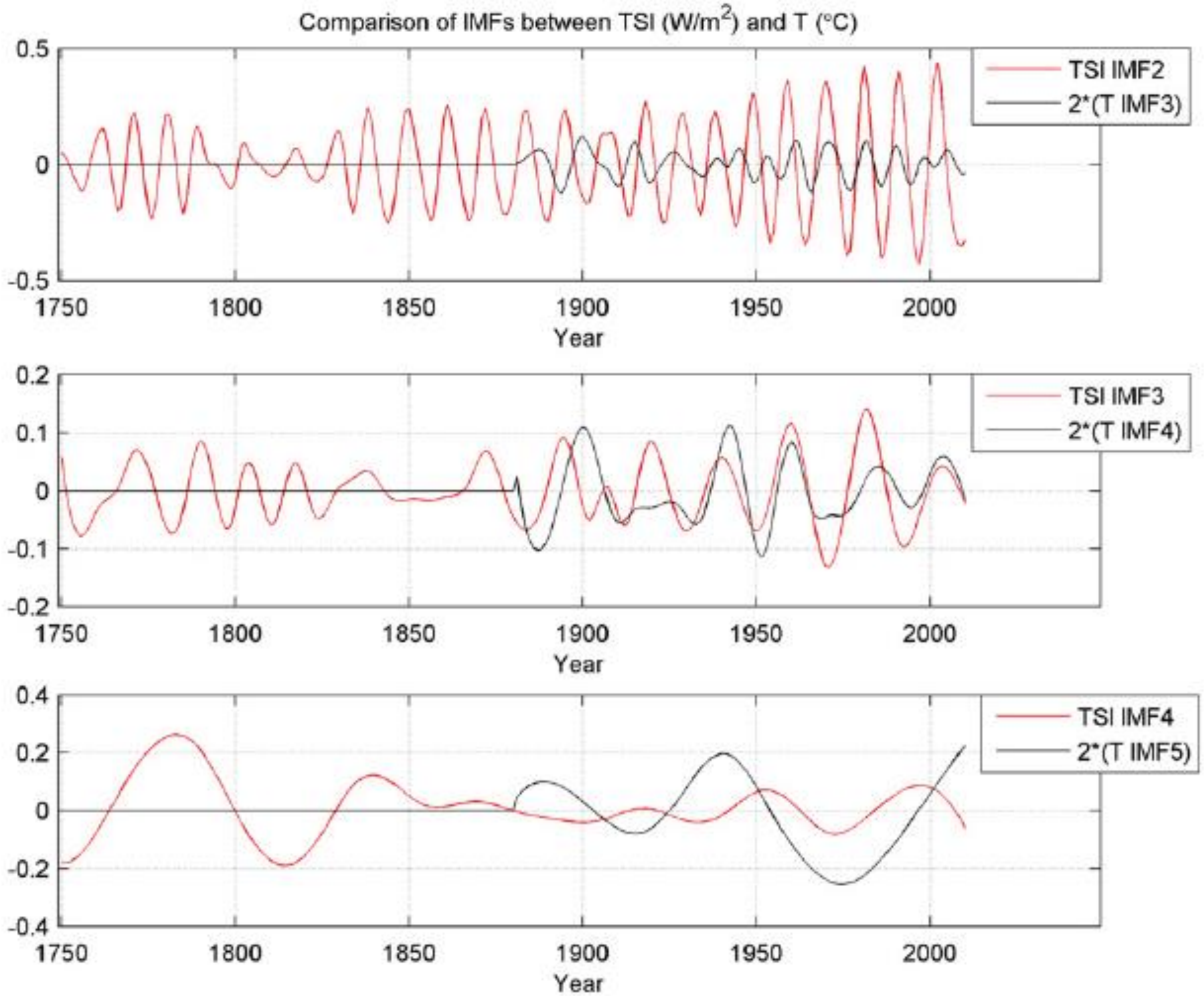


Subsection of IMF 2, the yearly cycle extracted from the CO_2 data using EEMD.

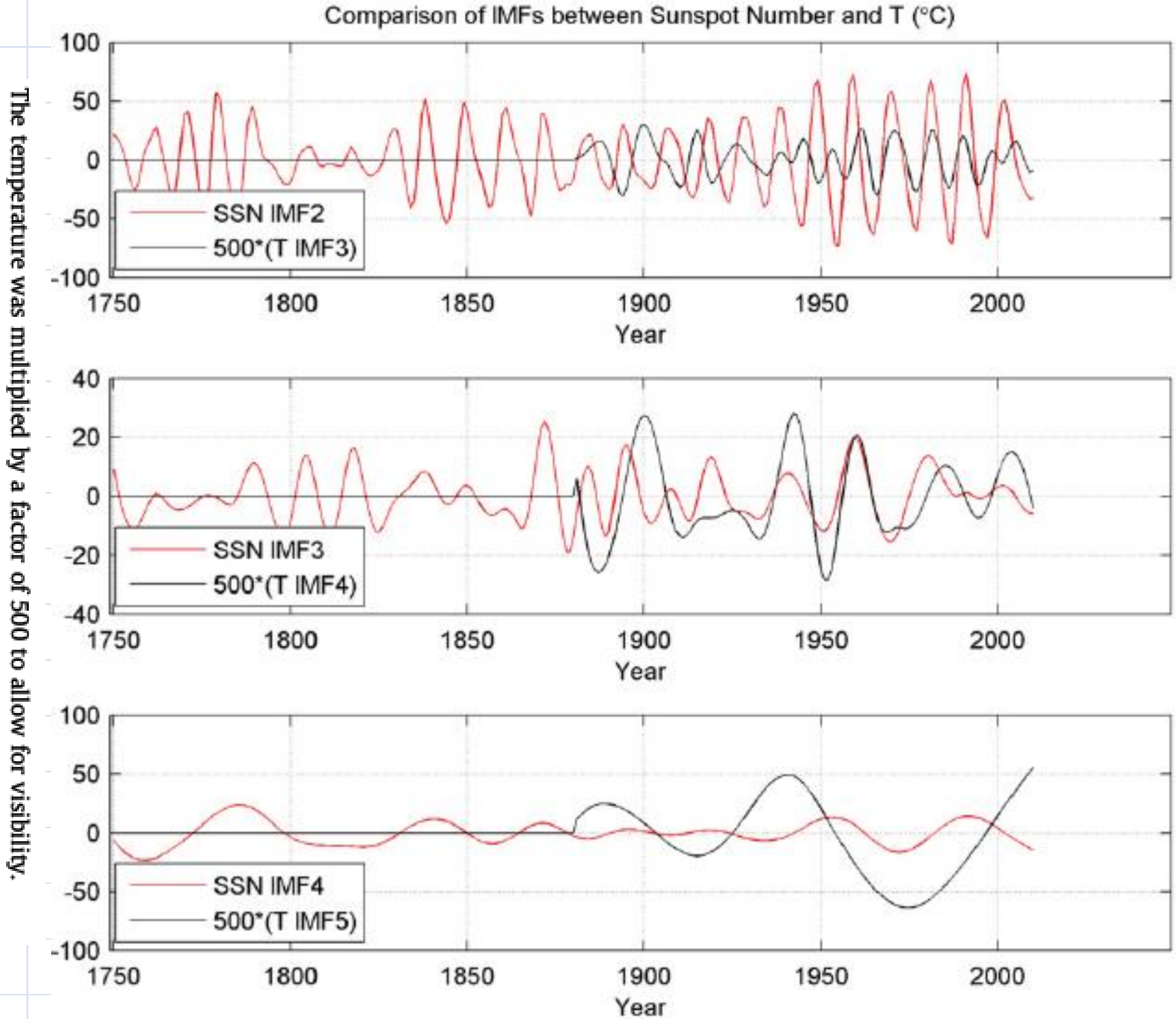
Comparison b/w TSI and sunspot number IMF



Comparison of IMFs for global mean temperature and TSI



Comparison of IMFs for global mean temperature and sunspot number data



Correlation Coeff. TSI and sunspot/GMT

Correlation coefficients (r)—Total Solar Irradiance and Sunspot from 1749 to 2009.

	Sunspot	IMF 1	IMF 2	IMF 3	IMF 4	IMF 5	IMF 6
TSI	**0.85**	0.50	0.82	0.27	0.28	0.33	0.20
IMF 1	0.21	**0.54**	0.26	0.04	−0.08	0.02	−0.02
IMF 2	0.61	0.54	**0.95**	0.10	−0.01	−0.001	−0.03
IMF 3	0.36	0.10	0.31	**0.74**	0.10	0.03	0.06
IMF 4	0.26	−0.02	0.01	0.19	**0.79**	−0.05	0.03
IMF 5	0.48	−0.01	0.03	0.01	0.31	**0.84**	0.25
IMF 6	0.59	−0.03	−0.05	−0.01	−0.001	0.59	**0.96**

Correlation coefficients (r)—Total Solar Irradiance and Global Mean Temperature from 1880 to 1945.

	TSI	IMF 1	IMF 2	IMF 3	IMF 4	IMF 5	IMF 6
T	0.28	0.02	−0.03	−0.34	0.13	0.28	0.41
IMF 1	−0.13	−0.03	−0.08	−0.09	−0.04	−0.04	−0.10
IMF 2	−0.18	0.004	−0.02	**−0.34**	−0.07	−0.04	−0.14
IMF 3	0.09	0.02	−0.02	−0.40	**0.30**	0.10	0.13
IMF 4	0.11	0.07	0.08	0.04	−0.06	**−0.26**	0.26
IMF 5	0.77	0.03	0.02	0.02	0.20	0.66	0.86
IMF 6	0.67	0.01	−0.04	−0.04	0.27	0.37	**0.87**

Correlation Coeff. GMT and TSI/sunspot

Correlation Coefficients (r)—Total Solar Irradiance and Global Mean Temperature from 1945 to 2009.

	TSI	IMF 1	IMF 2	IMF 3	IMF 4	IMF 5	IMF 6
T	0.13	0.01	0.09	0.39	0.27	−0.06	0.06
IMF 1	0.02	0.004	0.13	0.17	0.01	0.005	−0.04
IMF 2	−0.02	0.02	0.06	**0.43**	0.08	−0.10	−0.11
IMF 3	0.10	−0.003	0.03	0.08	**0.67**	−0.05	0.04
IMF 4	0.27	0.02	0.01	−0.05	0.13	**0.59**	0.08
IMF 5	−0.85	0.004	0.06	−0.006	−0.28	−0.81	−0.77
IMF 6	0.83	−0.04	−0.04	0.007	0.33	0.26	**0.99**

Correlation Coefficients (r)—Sunspot Number and Global Mean Temperature from 1880 to 1945.

	T	IMF 1	IMF 2	IMF 3	IMF 4	IMF 5	IMF 6
Sunspot #	0.04	0.02	−0.03	−0.34	0.06	0.17	0.07
IMF 1	−0.14	−0.09	−0.11	−0.02	−0.04	0.01	−0.13
IMF 2	−0.22	−0.005	−0.01	**−0.31**	−0.15	−0.06	−0.17
IMF 3	−0.03	0.08	−0.02	−0.37	**0.20**	0.08	−0.07
IMF 4	−0.16	0.06	0.01	0.01	0.36	**−0.42**	−0.22
IMF 5	0.79	0.04	0.03	0.05	0.28	0.80	0.76
IMF 6	0.68	0.01	−0.04	−0.03	0.27	0.40	**0.88**

Correlation Coeff. GMT and sunspot

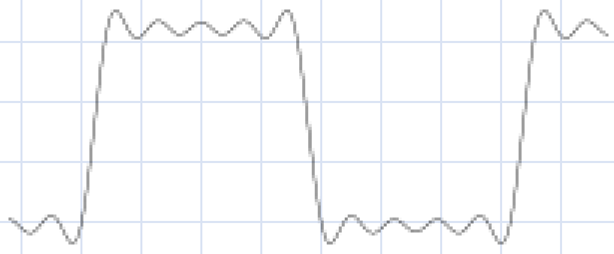
Correlation Coefficients (r)—Sunspot Number and Global Mean Temperature from 1945 to 2009.

	T	IMF 1	IMF 2	IMF 3	IMF 4	IMF 5	IMF 6
Sunspot #	−0.11	0.01	0.08	0.34	0.13	−0.14	−0.20
IMF 1	−0.13	−0.09	0.05	−0.05	−0.07	−0.11	−0.09
IMF 2	0.04	0.02	0.07	**0.42**	0.08	−0.03	−0.05
IMF 3	0.06	−0.005	0.02	0.09	**0.68**	−0.20	0.05
IMF 4	−0.02	0.03	0.02	−0.10	0.03	**0.30**	−0.17
IMF 5	−0.90	0.01	0.05	−0.005	−0.31	−0.64	−0.90
IMF 6	0.83	−0.04	−0.04	0.007	0.33	0.26	**0.99**

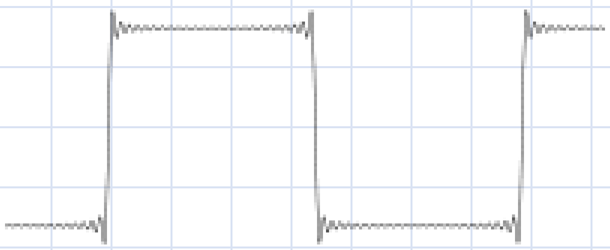
Gibbs Effect (.wiki)

- In non-continuous signal with an abrupt discontinuity, when finite number of Fourier coefficients was used, Gibbs phenomenon was occurred.
- All of 3 mentioned transforms suffers this effect.
- Wavelets are more useful for describing these signals with discontinuities because of their time-localized behavior (both Fourier and wavelet transforms are frequency-localized, but wavelets have an additional time-localization property). Because of this, many types of signals in practice may be non-sparse in the Fourier domain, but very sparse in the wavelet domain.
- For HHT could be used more sophisticated methods, such as Riesz basis, to avoid the Gibbs phenomenon.
- The reducing methods of this phenomenon for Fourier, were introduced in [Hamming 2003]

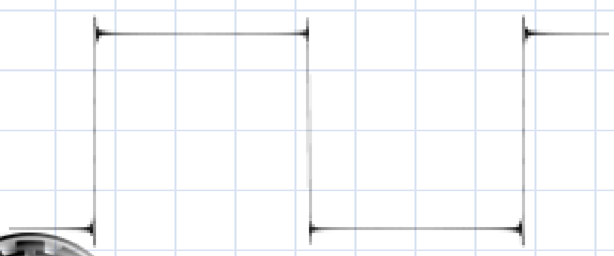
Example (.wiki)



Functional approximation of square wave using 5 harmonics



Functional approximation of square wave using 25 harmonics



Functional approximation of square wave using 125 harmonics

HHT implementation

- commercial software called the Hilbert-Huang transform data processing system (HHT-DPS) which was developed by Norden Huang at NASA and is available through NASA’s website.

- There are also publicly available Matlab codes by Patrick Flandrin and R code which extract IMFs from a given input data series.

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