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MODAL ANALYSIS OF NON-CLASSICALLY DAMPED LINEAR SYSTEMS

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SUMMARY

A critical, textbook-like review of the generalized modal superposition method of evaluating the dynamic response of non-classically damped linear systems is presented, which it is hoped will increase the attractiveness of the method to structural engineers and its application in structural engineering practice and research. Special attention is given to identifying the physical significance of the various elements of the solution and to simplifying its implementation. It is shown that the displacements of a non-classically damped n-degree-of-freedom system may be expressed as a linear combination of the displacements and velocities of n similarly excited single-degree-of-freedom systems, and that once the natural frequencies of vibration of the system have been determined, its response to an arbitrary excitation may be computed with only minimal computational effort beyond that required for the analysis of a classically damped system of the same size. The concepts involved are illustrated by a series of examples, and comprehensive numerical data for a three-degree-of-freedom system are presented which elucidate the effects of several important parameters. The exact solutions for the system are also compared over a wide range of conditions with those computed approximately considering the system to be classically damped, and the interrelationship of two sets of solutions is discussed.

INTRODUCTION

The modal superposition method is generally recognized as a powerful method for evaluating the dynamic response of viscously damped linear structural systems. The method enables one to express the response of a multi-degree-of-freedom system as a linear combination of its corresponding modal responses. Two versions of the procedure are in use: (1) the time history version, in which the modal responses are evaluated as a function of time and then combined to yield the response history of the system; and (2) the response spectrum version, in which first the maximum values of the modal responses are determined, usually from the response spectrum applicable to the particular excitation and damping under consideration, and the maximum response of the system is then computed by an appropriate combination of the modal maxima.

When damping is of the form specified by Caughey and O'Kelly,[1] the natural modes of vibration of the system are real-valued and identical to those of the associated undamped system. Systems satisfying this condition are said to be classically damped, and the modal superposition method for such systems is referred to as the classical modal method.

The classical modal method has found widespread application in civil engineering practice because of its conceptual simplicity, ease of application and the insight it provides into the action of the system. The response spectrum variant of the method, which makes it possible to identify and consider only the dominant terms in the solution, is particularly useful for making rapid estimates of maximum response values.

Viscously damped systems that do not satisfy the Caughey–O'Kelly condition generally have complex-valued natural modes. Such systems are said to be non-classically damped, and their response may be evaluated by a generalization of the modal superposition method due to Foss.[2]

Although well established,[2–9] the generalized modal method has found only limited application in structural engineering practice. Several factors appear to have contributed to this: (a) the generalized method is

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inherently more involved than the classical; (b) when used in conjunction with the response spectrum concept, it has had to rely on approximations of questionable accuracy; and (c) perhaps most important, the physical meanings of the elements of the solution for this method have not yet been identified as well as those for the classical modal method.

Considering that the specification of the nature and magnitude of damping in structures is subject to considerable uncertainty, the assumption of classical damping may be a justifiable approximation in many practical applications. There are instances, however, in which a more refined analysis is definitely warranted, and it is important that the attractiveness and reliability of the generalized procedure be improved. This paper is intended to be responsive to this need.

Its broad objectives are: (1) to simplify the implementation of the generalized modal method for evaluating the dynamic response of non-classically damped linear structures; and (2) by clarifying the physical significance of the elements of this method, to increase the attractiveness of the method to, and its use by, structural engineers.

The method is described by reference to cantilever structures excited at the base, and it is then applied to force-excited systems. Following an examination of the natural frequencies and modes of vibration of such systems, the free vibrational response is formulated in several alternative forms, and the more convenient form, in combination with the concepts employed in the development of Duhamel's integral for single-degree-of-freedom systems,[10] is used to obtain the solution for forced vibration. Novel features of the present treatment are: the physical interpretation of the solutions represented by complex conjugate pairs of characteristic vectors, the use of these solutions as building blocks in the definition of the free vibrational and forced vibrational responses of the system, the identification of some useful relationships for the vectors in the expression for forced vibration, and the analysis of systems for which some of the natural modes are real-valued and the others are complex.

It is shown that the transient displacements of a non-classically damped multi-degree-of-freedom system may be expressed as a linear combination of the deformations and the true relative velocities of a series of similarly excited single-degree-of-freedom systems, and that the analysis may be implemented with only minor computational effort beyond that required for a classically damped system of the same size. The concepts involved are illustrated by a series of examples, and comprehensive numerical solutions are presented which elucidate the sensitivity of the response to variations in several important parameters. The exact solutions also are compared with those obtained by an approximate modal superposition involving the use of classical modes of vibration,[11] and the interrelationship of the two sets of solutions is discussed.

Although use is made of complex-valued algebra in the derivation of some of the equations presented herein, all final expressions are presented in terms of real-valued quantities.

Notation

The symbols used in this paper are defined when first introduced in the text, and those used extensively are summarized in Appendix II.

STATEMENT OF PROBLEM

A viscously damped, linear cantilever system with n degrees of freedom is considered. The system is presumed to be excited by a base motion the acceleration of which is $\ddot{x}_g(t)$. The response of this system is governed by the equations

$$[m]\{\ddot{x}\}+[c]\{\dot{x}\}+[k]\{x\} = -[m]\{1\}\ddot{x}_g(t) \tag{1}$$

in which $\{x\}$ is the column vector of the displacements of the nodes relative to the moving base; a dot superscript denotes differentiation with respect to time, t; $\{1\}$ is a column vector of ones; and $[m]$, $[c]$ and $[k]$ are the mass matrix, damping matrix and stiffness matrix of the system, respectively. The latter matrices are real and symmetric; additionally, $[c]$ and $[k]$ are positive semi-definite, and $[m]$ is positive definite. The objective is to elucidate the analysis of the response of the system when no further restriction is imposed on the form of the damping matrix.

NATURAL FREQUENCIES AND MODES

For a system in free vibration, the right-hand member of equation (1) vanishes, and the equation admits a solution of the form

$$\{x\} = \{\psi\}e^{rt} \tag{2}$$

where r is a characteristic value and $\{\psi\}$ is the associated characteristic vector or natural mode. On substituting equation (2) into the homogeneous form of equation (1), one obtains the characteristic value problem

$$(r^2[m] + r[c] + [k])\{\psi\} = \{0\} \tag{3}$$

in which $\{0\}$ is the null vector.

Rather than through the solution of the system of equations (3), the desired values of r and the associated $\{\psi\}$ may be determined more conveniently by first reducing the system of n second order differential equations (1) to a system of $2n$ first order differential equations, as suggested in References 3 and 4. Summarized briefly in Appendix I, this approach leads to a characteristic value problem of the form

$$r[A]\{\dot{Z}\} + [B]\{Z\} = \{0\} \tag{4}$$

in which $[A]$ and $[B]$ are symmetric, real matrices of size $2n$ by $2n$; and $\{Z\}$ is a vector of $2n$ elements, of which the lower n elements represent the desired modal displacements, $\{\psi\}$, and the upper n elements represent the associated velocities, $r\{\psi\}$. Equation (4) may be solved by well established procedures.

Provided the amount of damping in the system is not very high, the characteristic values occur in complex conjugate pairs with either negative or zero real parts. For a system with n degrees of freedom, there are n pairs of characteristic values, and to each such pair there corresponds a complex conjugate pair of characteristic vectors.

Let r_j and $\bar{r}_j$ be a pair of characteristic values defined by

$$\left.\begin{matrix} r_j \\ \bar{r}_j \end{matrix}\right\} = -q_j \pm i\tilde{p}_j \tag{5}$$

and

$$\left.\begin{matrix} \{\psi_j\} \\ \{\bar{\psi}_j\} \end{matrix}\right\} = \{\phi_j\} \pm i\{\chi_j\} \tag{6}$$

be the associated pair of characteristic vectors. In these expressions, $i = \sqrt{-1}$; q_j and $\tilde{p}_j$ are real positive scalars; and $\{\phi_j\}$ and $\{\chi_j\}$ are real-valued vectors of n elements each. Further, let p_j be the modulus of r_j, i.e.

$$p_j = \sqrt{(q_j^2 + \tilde{p}_j^2)} \tag{7}$$

and

$$\zeta_j = \frac{q_j}{p_j} \tag{8}$$

Then the characteristic values may be expressed as

$$\left.\begin{matrix} r_j \\ \bar{r}_j \end{matrix}\right\} = -\zeta_j p_j \pm i\tilde{p}_j \tag{9}$$

and the following relationship between $\tilde{p}_j$ and p_j is determined from equation (7):

$$\tilde{p}_j = p_j\sqrt{(1 - \zeta_j^2)} \tag{10}$$

The values of p_j are numbered in ascending order, and the values of r_j and the associated vectors, $\{\psi_j\}$, are numbered in the order of the corresponding p_j.

Equations (9) are the same as those governing the characteristic roots of a viscously damped single-degree-of-freedom (SDF) system with an undamped circular natural frequency p_j and a damping factor ζ_j. Furthermore, equation (10) is the same as that relating the undamped and damped frequencies of such a system (see second section in Appendix I). In the following developments, p_j will be referred to as the jth pseudo-undamped circular natural frequency of the system; $\tilde{p}_j$ will be referred to as the corresponding damped frequency; and ζ_j as the jth modal damping factor. It should be noted that p_j is a function of the amount of system damping present and, hence, differs from the corresponding frequency of the associated undamped system. Where confusion may arise, the latter frequency will be denoted by the symbol p_j^0.

Reduction for undamped and classically damped systems

For an undamped system, the characteristic values r_j are purely imaginary and the natural modes are real-valued. Accordingly, $\{\chi_j\} = \{0\}$, $\{\psi_j\} = \{\bar{\psi}_j\} = \{\phi_j\}$, and $\tilde{p}_j = p_j = p_j^0$.

Caughey and O'Kelly[1] have shown that if the damping matrix of the system satisfies the identity

$$[c][m]^{-1}[k] = [k][m]^{-1}[c] \tag{11}$$

the natural modes are real-valued and equal to those of the associated undamped system. The exponent in this expression denotes the inverse of a matrix. The characteristic values in this case appear in complex conjugate pairs, and the modulus of each pair, p_j, is the same as the circular natural frequency of the associated undamped system, p_j^0. The Rayleigh form of damping,[12] for which $[c]$ is proportional to either $[m]$ or $[k]$ or is a linear combination of the two, is a special case of equation (11).

Orthogonality of modes

Any pair of characteristic vectors corresponding to distinct characteristic values, including a complex conjugate pair of modes, satisfies the following orthogonality relations:

$$(r_j + r_k)\{\psi_j\}^T[m]\{\psi_k\} + \{\psi_j\}^T[c]\{\psi_k\} = 0 \tag{12}$$

and

$$\{\psi_j\}^T[k]\{\psi_k\} - r_k r_j\{\psi_j\}^T[m]\{\psi_k\} = 0 \tag{13}$$

The derivation of these expressions is reviewed briefly in the third section of Appendix I.

For a classically damped system for which $\{\psi_j\} = \{\phi_j\}$ and $p_j = p_j^0$, it can be shown (see fourth section of Appendix I) that

$$\{\psi_j\}^T[c]\{\psi_k\} = 2\zeta_j p_j\{\phi_j\}^T[m]\{\phi_k\} \tag{14}$$

On substituting this equation into equation (12) and making use of the fact that the real and imaginary parts of the resulting expression must be zero, one obtains the well-known relation

$$\{\phi_j\}^T[m]\{\phi_k\} = 0 \quad \text{for } r_k \neq r_j \tag{15}$$

Furthermore, on substituting the latter equation into equation (13), one obtains

$$\{\phi_j\}^T[k]\{\phi_k\} = 0 \quad \text{for } r_k \neq r_j \tag{16}$$

It should be recalled that the quantities p_j and $\{\phi_j\}$ in these expressions are the same as those for the associated undamped system.

Examples

The sensitivity of the free vibrational characteristics of a non-classically damped system to the magnitude and distribution of the damping present are examined in this section for the three-storey building frame shown in Figure 1. The structure is presumed to be of the shear-beam type with uniform storey stiffnesses, k, and with floor masses m, m and $m/2$, as indicated. System damping is concentrated either in the bottom storey or the top storey. The damping coefficient, c, is expressed in the form

$$c = \zeta_0\sqrt{(km)} \tag{17}$$

and several different values of the dimensionless constant, ζ_0, are used. The damping matrices for these systems do not satisfy equation (11).

The characteristic values and vectors of these systems were evaluated by use of a standard computer program[13] making use of the procedure reviewed in the first section of Appendix I. The results for the system

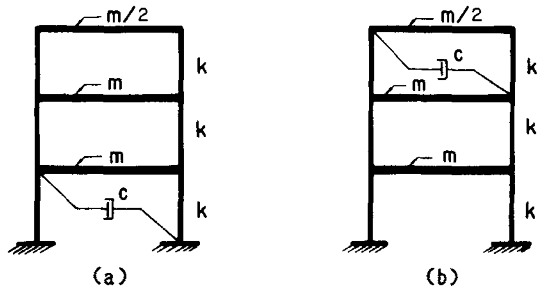

Figure 1. Systems considered

shown in Figure 1(a) are given in Table I for four different values of ζ_0. Listed in addition to the values of r_j and $\{\psi_j\}$ are the values of p_j, $\tilde{p}_j$ and ζ_j. The characteristic vectors are normalized such that the real part of the displacement of the first floor is unity and the corresponding imaginary part is zero.

The components $\{\phi_j\}$ and $\{\chi_j\}$ of the characteristic vectors for the system with the damper in the bottom storey are also plotted in Figure 2. Note that the imaginary component, $\{\chi_j\}$, may be quite substantial for high

Table I. Free vibrational characteristics of system shown in Figure 1(a)

Quantity	Floor level	First mode	Second mode	Third mode
			(a) For $\zeta_0 = 0$	
$r_j/\sqrt{(k/m)}$		0·5176i	1·4142i	1·9318i
$\{\psi_j\}$	1	1	1	1
	2	1·7321	0	−1·7321
	3	2	−1	2
$p_j^0/\sqrt{(k/m)}$		0·5176	1·4142	1·9318
			(b) For $\zeta_0 = 0{\cdot}1$	
$r_j/\sqrt{(k/m)}$		−0·0083 + 0·5178i	−0·0334 + 1·4138i	−0·0083 + 1·9311i
$\{\psi_j\}$	1	1	1	1
	2	1·7312 + 0·0431i	−0·0011 + 0·0470i	−1·7300 + 0·1611i
	3	1·9987 + 0·0598i	−0·9956 + 0·0002i	1·9968 − 0·2233i
$p_j/\sqrt{(k/m)}$		0·5179	1·4142	1·9312
$\tilde{p}_j/\sqrt{(k/m)}$		0·5178	1·4138	1·9311
ζ_j		0·0161	0·0236	0·0043
			(c) For $\zeta_0 = 0{\cdot}5$	
$r_j/\sqrt{(k/m)}$		−0·0420 + 0·5207i	−0·1724 + 1·4022i	−0·0356 + 1·9159i
$\{\psi_j\}$	1	1	1	1
	2	1·7096 + 0·2166i	−0·0225 + 0·2175i	−1·6871 + 0·8217i
	3	1·9681 + 0·3001i	−0·8963 + 0·0248i	1·9281 − 1·1419i
$p_j/\sqrt{(k/m)}$		0·5224	1·4127	1·9162
$\tilde{p}_j/\sqrt{(k/m)}$		0·5207	1·4022	1·9159
ζ_j		0·0804	0·1221	0·0186
			(d) For $\zeta_0 = 1{\cdot}0$	
$r_j/\sqrt{(k/m)}$		−0·0861 + 0·5313i	−0·3865 + 1·3425i	−0·0454 + 1·8870i
$\{\psi_j\}$	1	1	1	1
	2	1·6391 + 0·4398i	−0·0350 + 0·3531i	−1·6041 + 1·7155i
	3	1·8680 + 0·6089i	−0·6623 + 0·1524i	1·7944 − 2·3987i
$p_j/\sqrt{(k/m)}$		0·5382	1·3922	1·8875
$\tilde{p}_j/\sqrt{(k/m)}$		0·5313	1·3425	1·8870
ζ_j		0·1599	0·2647	0·0241

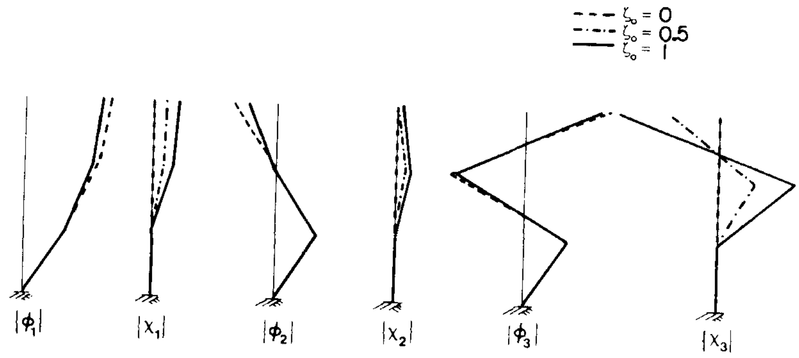

Figure 2. Effect of damping on natural modes of system considered in Figure 1(a)

damping values, particularly for the higher modes of vibration. By contrast, the real component, $\{\phi_j\}$, is not particularly sensitive to the amount of damping present.

The circular natural frequencies of the system for each of the two distributions of damping considered are plotted in Figure 3 as a function of ζ_0. As would be expected from equation (10), the damped frequencies are lower than the pseudo-undamped. However, the former frequencies are not necessarily lower than the true undamped natural frequencies of the system, p_j^0, (i.e. those corresponding to $\zeta_0 = 0$); and for a high order mode, they may well be lower than the corresponding frequencies of a lower order mode. As an example, note that $\tilde{p}_2$ in Figure 3(b) increases with increasing ζ_0, and that the values of $\tilde{p}_3$ for the high values of ζ_0 are less than the corresponding values of $\tilde{p}_2$. It should be recalled that it is the pseudo-undamped, rather than the damped, frequencies of the system that are numbered in ascending order.

The modal damping factors for the system, ζ_j, are displayed in Figure 4. Note that each of these factors is affected differently by a change in the overall damping of the system, and that an increase in system damping may increase or decrease the modal damping values.

Comparison with results of approximate procedure

Included in Figure 4 for purposes of comparison are the values of ζ_j computed on the assumption that the transformation which in the classical modal superposition method diagonalizes the stiffness matrix of the system also diagonalizes the damping matrix. The displacements of the system in this approximation are expressed as a linear combination of its classical, undamped natural modes of vibration, and in evaluating the

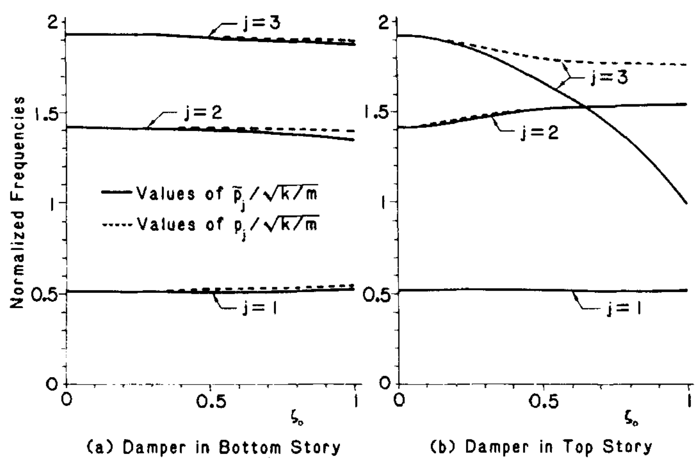

Figure 3. Effect of damping on natural frequencies of systems considered in Figure 1

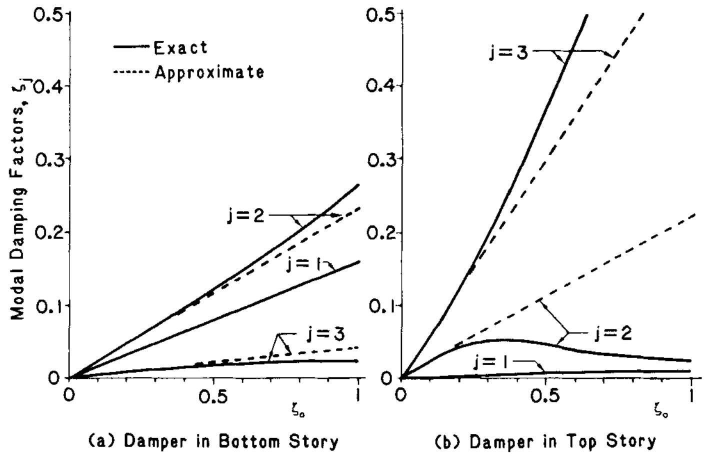

Figure 4. Effect of damping on modal damping factors for systems considered in Figure 1

triple matricial products $\{\phi_k\}^T[c]\{\phi_j\}$, the terms corresponding to $j \neq k$ are omitted. The superscript T in the latter expression denotes a transposed vector. This approach, which has been the subject of numerous previous studies,[11, 14-18] will be referred to herein as the approximate procedure. It may be seen that while the agreement between the two sets of results is generally reasonable, there are significant differences for the system considered in Figure 3(b), particularly for the larger values of ζ_0.

Selected values of the exact and approximate modal damping factors for the system in Figure 1(a) are listed in Table II, along with the associated natural frequency values, p_j and $\tilde{p}_j$.

MODAL SOLUTION

The term modal solution will be used to identify a solution represented by a linear combination of a complex conjugate pair of characteristic values and their associated vectors. In particular, the jth modal solution for displacements is given by

$$\{x\} = C_j\{\psi_j\}e^{r_j t} + \bar{C}_j\{\bar{\psi}_j\}e^{\bar{r}_j t} \tag{18}$$

in which C_j is a complex-valued constant and $\bar{C}_j$ is its complex conjugate.

Table II. Comparison of exact and approximate natural frequencies and damping factors for system considered in Figure 1(a)

Quantity	First mode Exact	First mode Approx.	Second mode Exact	Second mode Approx.	Third mode Exact	Third mode Approx.
			(a) $\zeta_0 = 0{\cdot}1$			
$p_j/\sqrt{(k/m)}$	0·5179	0·5176	1·4142	1·4142	1·9312	1·9318
$\tilde{p}_j/\sqrt{(k/m)}$	0·5178	0·5175	1·4138	1·4138	1·9311	1·9318
ζ_j	0·0161	0·0161	0·0236	0·0236	0·0043	0·0043
			(b) $\zeta_0 = 0{\cdot}5$			
$p_j/\sqrt{(k/m)}$	0·5224	0·5176	1·4127	1·4142	1·9162	1·9318
$\tilde{p}_j/\sqrt{(k/m)}$	0·5207	0·5159	1·4022	1·4044	1·9159	1·9314
ζ_j	0·0804	0·0805	0·1221	0·1179	0·0186	0·0216
			(c) $\zeta_0 = 1{\cdot}0$			
$p_j/\sqrt{(k/m)}$	0·5382	0·5176	1·3922	1·4142	1·8876	1·9318
$\tilde{p}_j/\sqrt{(k/m)}$	0·5313	0·5108	1·3425	1·3744	1·8870	1·9300
ζ_j	0·1599	0·1610	0·2647	0·2357	0·0241	0·0431

Since the second term on the right-hand member of equation (18) is the complex conjugate of the first, the sum of the imaginary terms in this expression vanishes and equation (18) reduces to

$$\{x\} = 2\,\mathrm{Re}[C_j\{\psi_j\}e^{r_j t}] \tag{19}$$

in which Re stands for the real part of the quantity that follows.

It is instructive to express equation (19) entirely in real-valued terms, and to this end two alternative forms are considered. If one expresses C_j in terms of its modulus and a phase angle as

$$C_j = \tfrac{1}{2}a_j e^{i\theta_j} \tag{20}$$

and makes use of equations (6) and (9), and of the identity between exponential and trigonometric functions, equation (19) may be written as

$$\{x\} = a_j e^{-\zeta_j p_j t}[\{\phi_j\}\cos(\tilde{p}_j t + \theta_j) - \{\chi_j\}\sin(\tilde{p}_j t + \theta_j)] \tag{21}$$

Alternatively, if one first evaluates the product

$$2C_j\{\psi_j\} = \{\beta_j\} + i\{\gamma_j\} \tag{22}$$

in which $\{\beta_j\}$ and $\{\gamma_j\}$ are real-valued vectors, then substitutes equation (22) into equation (19), and makes use of the identity between exponential and trigonometric functions, one obtains

$$\{x\} = e^{-\zeta_j p_j t}[\{\beta_j\}\cos\tilde{p}_j t - \{\gamma_j\}\sin\tilde{p}_j t] \tag{23}$$

Equations (19), (21) and (23) represent the superposition of two exponentially decaying harmonic motions with a circular frequency, $\tilde{p}_j$, and a damping factor, ζ_j. The component motions lag one another by 90 degrees or one-quarter the period $\tilde{T}_j = 2\pi/\tilde{p}_j$, and they are in different configurations. As a result, each point of the system undergoes a simple harmonic motion, but the configuration of the system does not remain constant but changes continuously, repeating itself at intervals $\tilde{T}_j$. The quantity $\tilde{T}_j$ is known as the jth damped natural period of the system.

It should be clear from equation (21) that the $\{\phi_j\}$ configuration is attained when $\cos(\tilde{p}_j t + \theta_j) = \pm 1$ or $\sin(\tilde{p}_j t + \theta_j) = 0$, whereas the $\{\chi_j\}$ configuration is attained when $\sin(\tilde{p}_j t + \theta_j) = \pm 1$ or $\cos(\tilde{p}_j t + \theta_j) = 0$. It can further be seen from equation (23) that the $\{\beta_j\}$ configuration is attained when $\cos\tilde{p}_j t = \pm 1$ or $\sin\tilde{p}_j t = 0$, whereas the $\{\gamma_j\}$ configuration is attained when $\sin\tilde{p}_j t = \pm 1$ or $\cos\tilde{p}_j t = 0$.

It is important to realize that equations (21) and (23) are only two of an infinite number of forms in which the modal solution may be expressed. Other combinations of the basic modal components $\{\phi_j\}$ and $\{\chi_j\}$ can be used as the reference configurations, and this fact is used to advantage in a subsequent development.

Reduction for classically damped systems

For undamped systems, for which $\{\psi_j\} = \{\bar{\psi}_j\} = \{\phi_j\}$ and $\tilde{p}_j = p_j = p_j^0$, equation (21) reduces to

$$\{x\} = a_j\{\phi_j\}\cos(p_j^0 t + \theta_j) \tag{24}$$

Such systems can execute simple harmonic motions in fixed, time-invariant configurations.

For damped systems satisfying equation (11), equation (21) reduces to

$$\{x\} = a_j\{\phi_j\}e^{-\zeta_j p_j^0 t}\cos(\tilde{p}_j^0 t + \theta_j) \tag{25}$$

in which $\tilde{p}_j^0$ is related to p_j^0 by the same expression as that relating $\tilde{p}_j$ to p_j. Such systems can vibrate in time-invariant configurations, but with exponentially decaying amplitudes.

FREE VIBRATION

The response of the system to an arbitrary initial excitation is given by the superposition of the modal solutions presented in the preceding section. In particular, the displacements $\{x\}$ may be expressed either in the form of equation (19) as

$$\{x\} = 2\sum_{j=1}^{n}\mathrm{Re}[C_j\{\psi_j\}e^{r_j t}] \tag{26}$$

or in the form of equations (21) and (23) as

$$\{x\} = \sum_{j=1}^{n} a_j e^{-\zeta_j p_j t}[\{\phi_j\}\cos(\tilde{p}_j t + \theta_j) - \{\chi_j\}\sin(\tilde{p}_j t + \theta_j)] \quad (27)$$

or

$$\{x\} = \sum_{j=1}^{n} e^{-\zeta_j p_j t}[\{\beta_j\}\cos\tilde{p}_j t - \{\gamma_j\}\sin\tilde{p}_j t] \quad (28)$$

The complex-valued participation factors, C_j, may be determined from

$$C_j = \frac{r_j\{\psi_j\}^T[m]\{x(0)\} + \{\psi_j\}^T[c]\{x(0)\} + \{\psi_j\}^T[m]\{\dot{x}(0)\}}{2r_j\{\psi_j\}^T[m]\{\psi_j\} + \{\psi_j\}^T[c]\{\psi_j\}} \quad (29)$$

in which $\{x(0)\}$ is the prescribed vector of initial displacements and $\{\dot{x}(0)\}$ is the corresponding vector of initial velocities. The derivation of this equation is given under the fifth heading in Appendix I. With the values of C_j established, the constants a_j and θ_j in equation (27) are determined from equation (20), and the vectors $\{\beta_j\}$ and $\{\gamma_j\}$ in equation (28) are determined from equation (22).

For a classically damped system, the following generalized version of equation (14) is valid (see fourth section in Appendix I):

$$\{\psi_j\}^T[c]\{x(0)\} = 2\zeta_j p_j^0\{\phi_j\}^T[m]\{x(0)\} \quad (30)$$

and on making use of this result and of equations (9) and (14), equation (29) reduces to

$$C_j = \frac{1}{2}\left\{x_j^* - i\left[\frac{v_j^*}{\tilde{p}_j^0} + \frac{\zeta_j}{\sqrt{(1-\zeta_j^2)}}x_j^*\right]\right\} \quad (31)$$

in which

$$x_j^* = \frac{\{\phi_j\}^T[m]\{x(0)\}}{\{\phi_j\}^T[m]\{\phi_j\}} \quad (32)$$

and

$$v_j^* = \frac{\{\phi_j\}^T[m]\{\dot{x}(0)\}}{\{\phi_j\}^T[m]\{\phi_j\}} \quad (33)$$

Similarly, the vectors $\{\beta_j\}$ and $\{\gamma_j\}$ in equations (23) and (28) reduce to the well-known expressions:[10]

$$\{\beta_j\} = x_j^*\{\phi_j\} \quad (34a)$$

and

$$\{\gamma_j\} = -\left[\frac{v_j^*}{\tilde{p}_j^0} + \frac{\zeta_j}{\sqrt{(1-\zeta_j^2)}}x_j^*\right]\{\phi_j\} \quad (34b)$$

in which $\{\phi_j\}$ should be interpreted as the jth real-valued mode of the associated undamped system.

Equations (26) and (28) are the more convenient of the three forms used to express the response, and will be emphasized in the material that follows.

Initial conditions that excite a single mode

If the initial displacements and velocities of the system are of the form

$$\{x(0)\} = 2\,\mathrm{Re}[C_k\{\psi_k\}] \quad (35a)$$

and

$$\{\dot{x}(0)\} = 2\,\mathrm{Re}[r_k C_k\{\psi_k\}] \quad (35b)$$

it can be shown (see sixth section in Appendix I) that all values of C_j in equation (26), except for the kth, vanish, and that the response of the system is given by

$$\{x\} = 2\,\mathrm{Re}[C_k\{\psi_k\}e^{r_k t}] \quad (36)$$

To clarify the meaning of equations (35), let

$$2C_k = b_k + id_k \quad (37)$$

in which b_k and d_k are real-valued constants. On substituting this expression into equations (36) and making use of equation (9), one obtains

$$\{x(0)\} = b_k\{\phi_k\} - d_k\{\chi_k\} \tag{38a}$$

$$\{\dot{x}(0)\} = b'_k\{\phi_k\} - d'_k\{\chi_k\} \tag{38b}$$

in which

$$b'_k = -\tilde{p}_k\left[\frac{\zeta_k}{\sqrt{(1-\zeta_k^2)}}b_k + d_k\right] \tag{39a}$$

and

$$d'_k = \tilde{p}_k\left[b_k - \frac{\zeta_k}{\sqrt{(1-\zeta_k^2)}}d_k\right] \tag{39b}$$

It should now be clear that any initial displacement configuration which is a linear combination of $\{\phi_k\}$ and $\{\chi_k\}$, along with an initial velocity configuration defined by equations (38b) and (39), will excite only the kth mode of vibration of the system.

In particular, if $\{x(0)\} = b_k\{\phi_k\}$, the initial velocities needed to excite only the kth mode of vibration are determined from equations (38b) and (39) to be

$$\{\dot{x}(0)\} = -b_k(\zeta_k p_k\{\phi_k\} + \tilde{p}_k\{\chi_k\}) \tag{40}$$

Similarly, if $\{\dot{x}(0)\} = d'_k\{\chi_k\}$, the corresponding initial displacements are determined from equations (38a) by first computing the values of b_k and d_k from the system equations (39). The result is

$$\{x(0)\} = \frac{d'_k}{p_k}[\sqrt{(1-\zeta_k^2)}\{\phi_k\} - \zeta_k\{\chi_k\}] \tag{41}$$

With the proportionality factors b_k and d_k specified, the values of $2C_k$ may be determined from equation (37), and the displacements of the system at any time may be determined from equation (36). Alternatively, the displacements may be expressed directly in terms of b_k and d_k as follows:

$$\{x\} = e^{-\zeta_k p_k t}[(b_k\{\phi_k\} - d_k\{\chi_k\})\cos\tilde{p}_k t - (d_k\{\phi_k\} + b_k\{\chi_k\})\sin\tilde{p}_k t] \tag{42}$$

Free vibration due to uniform set of initial velocity changes

Before proceeding to the analysis of the forced response of the system, it is desirable to re-examine the response of the system to a uniform set of initial velocity changes with no corresponding displacement changes, i.e. $\{x(0)\} = \{0\}$ and $\{\dot{x}(0)\} = \{1\}v_0$.

Let B_j be the value of C_j for a unitary set of initial velocity changes. This value is determined from equation (29) to be

$$B_j = \frac{\{\psi_j\}^{\mathrm{T}}[m]\{1\}}{2r_j\{\psi_j\}^{\mathrm{T}}[m]\{\psi_j\} + \{\psi_j\}^{\mathrm{T}}[c]\{\psi_j\}} \tag{43}$$

The value of C_j for $\{\dot{x}(0)\} = \{1\}v_0$ is then $C_j = B_j v_0$, and the displacements of the system may be determined from the following expression deduced from equation (26):

$$\{x\} = 2\sum_{j=1}^{n} \mathrm{Re}[B_j\{\psi_j\}v_0 e^{r_j t}] \tag{44}$$

If the product $2B_j\{\psi_j\}$ is now expressed in a form analogous to equation (22) as

$$2B_j\{\psi_j\} = \{\beta_j^{\mathrm{v}}\} + \mathrm{i}\{\gamma_j^{\mathrm{v}}\} \tag{45}$$

in which $\{\beta_j^{\mathrm{v}}\}$ and $\{\gamma_j^{\mathrm{v}}\}$ are real-valued vectors with units of time per radian, equation (44) may also be written in the form

$$\{x\} = \sum_{j=1}^{n} e^{-\zeta_j p_j t}[\{\beta_j^{\mathrm{v}}\}\cos\tilde{p}_j t - \{\gamma_j^{\mathrm{v}}\}\sin\tilde{p}_j t]\,v_0 \tag{46}$$

The significance of the superscripts v in the last two expressions is identified later.

A simple but crucial final step will now be taken. Let $h_j(t)$ be the impulse response function for a SDF system, defined as the response of the system to a unit initial velocity change with no corresponding displacement change. For a viscously damped system with damping factor ζ_j and undamped circular natural frequency p_j, this function is given by

$$h_j(t) = \frac{1}{\tilde{p}_j} e^{-\zeta_j p_j t} \sin \tilde{p}_j t \tag{47}$$

and its first derivative is given by

$$\dot{h}_j(t) = e^{-\zeta_j p_j t}\left[\cos \tilde{p}_j t - \frac{\zeta_j}{\sqrt{(1-\zeta_j^2)}} \sin \tilde{p}_j t\right] \tag{48}$$

from which, on making use of equation (10), one obtains

$$e^{-\zeta_j p_j t} \cos \tilde{p}_j t = \dot{h}_j(t) + \zeta_j p_j h_j(t) \tag{49}$$

The time functions multiplying the vectors $\{\beta_j^{\mathrm{v}}\}$ and $\{\gamma_j^{\mathrm{v}}\}$ in equation (46) are now replaced by the corresponding expressions defined by equations (47) and (49) to obtain

$$\{x\} = \sum_{j=1}^{n} [\{\alpha_j^{\mathrm{v}}\} p_j h_j(t) + \{\beta_j^{\mathrm{v}}\} \dot{h}_j(t)] v_0 \tag{50}$$

in which

$$\{\alpha_j^{\mathrm{v}}\} = \zeta_j \{\beta_j^{\mathrm{v}}\} - \sqrt{(1-\zeta_j^2)}\{\gamma_j^{\mathrm{v}}\} \tag{51}$$

Arrived at independently in the course of this study, this transformation has also been used recently by Igusa *et al.*[19]

Equation (50) is analogous to equation (46) but it differs from the latter in two respects: (a) instead of the sine and cosine functions, it is expressed in terms of the functions $h_j(t)$ and $\dot{h}_j(t)$ which have clear physical meanings; and (b) the reference configurations are the vectors $\{\alpha_j^{\mathrm{v}}\}$ and $\{\beta_j^{\mathrm{v}}\}$ instead of the vectors $\{\beta_j^{\mathrm{v}}\}$ and $\{\gamma_j^{\mathrm{v}}\}$. In a modal solution, the configuration $\{\alpha_j^{\mathrm{v}}\}$ is attained at the instant for which $h_j(t)$ is an extremum, i.e. $\dot{h}_j(t) = 0$, whereas the configuration $\{\beta_j^{\mathrm{v}}\}$ is attained when $h_j(t)$ is zero. Equation (50) is fundamental to the analysis of the transient response that follows.

FORCED VIBRATION

The response of the system to an arbitrary excitation of the base may be evaluated from the expressions for free vibration presented in the preceding section as follows. If $\ddot{x}_g(\tau)$ is the acceleration of the base at time $t = \tau$, the velocity change of the base in the short time interval between τ and $\tau + d\tau$ is given by $\ddot{x}_g(\tau)d\tau$, and the velocity change of each mass of the structure relative to the moving base is given by $v(\tau) = -\ddot{x}_g(\tau)d\tau$.

The differential displacements of the system, $\{dx\}$, at $t > \tau$ due to these velocity changes are then given by the following expression, obtained from equation (50) by replacing v_0 with $v(\tau)$ and t with $t-\tau$:

$$\{dx\} = -\sum_{j=1}^{n} [\{\alpha_j^{\mathrm{v}}\} p_j h_j(t-\tau) + \{\beta_j^{\mathrm{v}}\} \dot{h}_j(t-\tau)] \ddot{x}_g(\tau) d\tau \tag{52}$$

The displacements of the system due to the prescribed base motion are finally obtained by integration as

$$\{x\} = \sum_{j=1}^{n} [\{\alpha_j^{\mathrm{v}}\} V_j(t) + \{\beta_j^{\mathrm{v}}\} \dot{D}_j(t)] \tag{53}$$

in which

$$V_j(t) = p_j D_j(t) = -p_j \int_0^t \ddot{x}_g(\tau) h_j(t-\tau) d\tau \tag{54}$$

and

$$\dot{D}_j(t) = -\int_0^t \ddot{x}_g(\tau) \dot{h}_j(t-\tau) d\tau \tag{55}$$

The quantity $V_j(t)$ in equations (53) and (54) represents the instantaneous pseudovelocity of a SDF system with circular frequency p_j and damping factor ζ_j subjected to the prescribed excitation; and $D_j(t)$ and $\dot{D}_j(t)$ represent the corresponding deformation and relative velocity of the system, respectively. It follows that the response of a non-classically damped multi-degree-of-freedom system may be expressed as a linear combination of n pairs of terms. The first member of the jth such pair represents a motion in a configuration $\{\alpha_j^{\text{v}}\}$ the temporal variation of which is the same as that of $D_j(t)$, whereas the second member represents a motion in a configuration $\{\beta_j^{\text{v}}\}$ the temporal variation of which is the same as that of $\dot{D}_j(t)$. The configurations $\{\alpha_j^{\text{v}}\}$ and $\{\beta_j^{\text{v}}\}$ are naturally functions of the natural modes of vibration of the system and are defined by equations (45) and (51).

In a stepwise numerical evaluation of the response of a SDF system, the relative velocity, $\dot{D}_j(t)$, is normally computed in the process of obtaining $D_j(t)$ or the associated pseudovelocity value, $V_j(t)$. Provided the natural frequencies and modes of a non-classically damped multi-degree-of-freedom system have been evaluated, therefore, the analysis of such a system may be implemented with only minor computational effort beyond that required for a classically damped system of the same size.

Alternative forms of expressions for response

Instead of the pseudovelocity and true relative velocity functions, $V_j(t)$ and $\dot{D}_j(t)$, equation (53) may also be expressed in terms of $D_j(t)$ and $\dot{D}_j(t)$, as follows.

Let $\{\alpha_j^{\text{D}}\}$ and $\{\beta_j^{\text{D}}\}$ be dimensionless vectors defined by

$$\{\alpha_j^{\text{D}}\} = p_j\{\alpha_j^{\text{v}}\} \tag{56a}$$

and

$$\{\beta_j^{\text{D}}\} = p_j\{\beta_j^{\text{v}}\} \tag{56b}$$

On making use of the relationship between $V_j(t)$ and $D_j(t)$ defined by equation (54), equation (53) may then be rewritten as

$$\{x\} = \sum_{j=1}^{n} \left[\{\alpha_j^{\text{D}}\} D_j(t) + \{\beta_j^{\text{D}}\} \frac{\dot{D}_j(t)}{p_j} \right] \tag{57}$$

Similarly, on introducing the vectors

$$\{\alpha_j^{\text{A}}\} = \frac{1}{p_j}\{\alpha_j^{\text{v}}\} = \frac{1}{p_j^2}\{\alpha_j^{\text{D}}\} \tag{58a}$$

and

$$\{\beta_j^{\text{A}}\} = \frac{1}{p_j}\{\beta_j^{\text{v}}\} = \frac{1}{p_j^2}\{\beta_j^{\text{D}}\} \tag{58b}$$

and the pseudo-acceleration function, $A_j(t)$, defined by

$$A_j(t) = p_j V_j(t) = p_j^2 D_j(t) \tag{59}$$

equation (53) can also be written as

$$\{x\} = \sum_{j=1}^{n} [\{\alpha_j^{\text{A}}\} A_j(t) + \{\beta_j^{\text{A}}\} p_j \dot{D}_j(t)] \tag{60}$$

Effect of non-zero initial conditions

Implicit in the foregoing development has been the assumption that the system is initially at rest. For a system with non-zero initial conditions, equation (53), or its equivalent versions defined by equations (57) and (60), should be augmented by the addition of the free vibrational solution defined by equations (26), (27) or (28).

Reduction for classically damped systems

For classically damped systems, for which the $p_j = p_j^0$ and the natural modes of vibration are real-valued and satisfy equations (15) and (16), equation (43) reduces to $B_j = -ib_j^{\text{v}}/[2\sqrt{(1-\zeta_j^2)}]$, in which

$$b_j^{\mathrm{v}} = \frac{1}{p_j^{\mathrm{o}}} \frac{\{\phi_j\}^{\mathrm{T}}[m]\{1\}}{\{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\}} \tag{61}$$

From equations (45) and (51) it then follows that $\{\beta_j^{\mathrm{v}}\} = \{0\}$ and that $\{\alpha_j^{\mathrm{v}}\} = b_j^{\mathrm{v}}\{\phi_j\}$. Thus, equation (53) reduces to the well-known expression

$$\{x\} = \sum_{j=1}^{n} b_j^{\mathrm{v}}\{\phi_j\}\, V_j(t) \tag{62}$$

The v-superscript on the symbol b_j emphasizes the fact that the latter quantity is to be used along with the pseudovelocity function, $V_j(t)$.

Equation (62) can also be expressed in terms of the deformation function, $D_j(t)$, as

$$\{x\} = \sum_{j=1}^{n} b_j^{\mathrm{D}}\{\phi_j\}\, D_j(\mathrm{t}) \tag{63}$$

or in terms of the pseudo-acceleration function, $A_j(t)$, as

$$\{x\} = \sum_{j=1}^{n} b_j^{\mathrm{A}}\{\phi_j\}\, A_j(t) \tag{64}$$

in which b_j^{D} and b_j^{A} are participation factors defined by

$$b_j^{\mathrm{D}} = \frac{\{\phi_j\}^{\mathrm{T}}[m]\{1\}}{\{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\}} \tag{65a}$$

and

$$b_j^{\mathrm{A}} = \frac{1}{(p_j^{\mathrm{o}})^2} \frac{\{\phi_j\}^{\mathrm{T}}[m]\{1\}}{\{\phi_j\}^{\mathrm{T}}[m]\{\phi_j\}} \tag{65b}$$

Summary of procedure

The steps involved in the analysis of the transient response of a non-classically damped system may be summarized as follows.

1. Evaluate the characteristic values, r_j, and the associated characteristic vectors, $\{\psi_j\}$; and from equations (5), (7) and (8), determine the damped and pseudo-undamped natural frequencies of the system, $\tilde{p}_j$ and p_j, and the modal damping factors, ζ_j.
2. From equation (43), compute the participation factors, B_j.
3. Evaluate the complex-valued products $2B_j\{\psi_j\} = \{\beta_j^{\mathrm{v}}\} + \mathrm{i}\{\gamma_j^{\mathrm{v}}\}$, and by application of equation (51), compute the vectors $\{\alpha_j^{\mathrm{v}}\}$. Alternatively, one may compute the vectors $\{\alpha_j^{\mathrm{D}}\}$ and $\{\beta_j^{\mathrm{D}}\}$ from equations (56) or the vectors $\{\alpha_j^{\mathrm{A}}\}$ and $\{\beta_j^{\mathrm{A}}\}$ from equations (58).
4. From analyses of the response of single-degree-of-freedom systems to the prescribed ground motion, determine the pseudovelocity functions, $V_j(t)$, and the true relative velocities, $\dot{D}_j(t)$.
5. Compute the displacements $\{x\}$ from equation (53), and the corresponding storey deformations from

$$u_i = x_i - x_{i-1} \tag{66}$$

in which the subscript i refers to the ith floor level or storey. The displacements may also be computed from equation (57) or from equation (60).

Properties of modal response vectors

The vectors $\{\alpha_j\}$ and $\{\beta_j\}$ with the various superscripts in equations (53), (57) and (60) satisfy the following relations:

$$\sum_{j=1}^{n} \{\beta_j^{\mathrm{v}}\} = \{0\} \tag{67a}$$

$$\sum_{j=1}^{n} [\{\alpha_j^D\} - 2\zeta_j\{\beta_j^D\}] = \{1\} \tag{67b}$$

and

$$-\ddot{x}_g \sum_{j=1}^{n} \{\alpha_j^A\} = \{x_{st}\} \tag{67c}$$

in which $\{x_{st}\}$ represents the static displacements of the structure due to the inertia forces associated with a uniform structural acceleration of magnitude $\ddot{x}_g$. These expressions are of great value in checking the accuracy of the solution.

Equation (67a) is deduced from equations (44) and (45) on noting that, for the conditions considered, $\{x(0)\} = \{0\}$. Similarly, equation (67b) is obtained from the first derivative of equation (44) by making use of the fact that $\{\dot{x}(0)\} = \{1\}$. Finally, equation (67c) is obtained by examining the high-frequency limiting behaviour of equation (60). For very stiff systems, the maximum values of $\{x\}$ tend to $\{x_{st}\}$; the corresponding values of $A_j(t)$ and $\dot{D}_j(t)$ tend to $-\ddot{x}_g$ and zero, respectively; and equation (60) leads to equation (67c).

Illustrative example

The response of the system shown in Figure 1(a) is evaluated in this section for two different excitations of the base: (a) the half-cycle displacement pulse shown in Figure 5, for which the acceleration trace consists of a sequence of three half-sine waves of the same peak values and durations t_1, $2t_1$ and t_1, respectively; and (b) the first 6·3 s of the N–S component of the El Centro, California earthquake record of May 18, 1940, as reported in Reference 20. The peak values of the acceleration, velocity and displacement of the latter motion are $\ddot{x}_g = 0{\cdot}312g$, $\dot{x}_g = 14{\cdot}02$ in/s and $x_g = 8{\cdot}29$ in, respectively. The dimensionless damping coefficient in equation (17) is assigned the values of $\zeta_0 = 0{\cdot}5$ and $\zeta_0 = 1$. The values of r_j and $\{\psi_j\}$ for these systems are listed in Table I along with the associated values of p_j, $\tilde{p}_j$ and ζ_j.

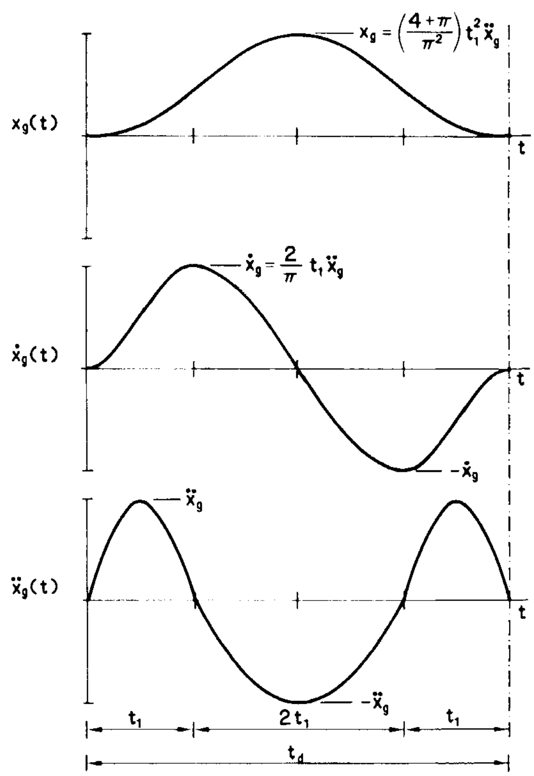

Figure 5. Simple base motion considered

The complex-valued participation factors B_j are determined from equation (43), and they are listed in Table III along with the products $2B_j\{\psi_j\} = \{\beta_j^{\mathrm{y}}\} + \mathrm{i}\{\gamma_j^{\mathrm{y}}\}$ and the associated vectors $\{\alpha_j^{\mathrm{y}}\}$, $\{\alpha_j^{\mathrm{P}}\}$, $\{\beta_j^{\mathrm{P}}\}$, $\{\alpha_j^{\mathrm{A}}\}$, and $\{\beta_j^{\mathrm{A}}\}$. These results, which are independent of the characteristics of the base motion, can be shown to satisfy equations (67). The common multipliers for the various quantities are identified in the extreme right-hand column of the table.

With the information presented in Table III, the displacements of the system may be determined from any one of equations (53), (57) or (60), making use of the relevant SDF response functions, $D_j(t)$ and $\dot{D}_j(t)$. The interfloor deformations may then be determined from equation (66).

The histories of storey deformation for systems subjected to the half-cycle displacement pulse are shown by the solid lines in Figures 6 and 7. Two different values of the frequency parameter $f_1^0 t_{\mathrm{d}}$ are considered, in which $f_1^0 = p_1^0/2\pi$ is the fundamental undamped natural frequency of the system in cycles per unit of time, and t_{d} is the duration of the forcing function. Similar data are presented in Figure 8 for systems with $f_1^0 = 1$ cps subjected to the El Centro earthquake record. In each case, the response of the system is displayed

Table III. Response of system considered in Figure 1(a) due to a base excitation

Floor level	First mode		Second mode		Third mode		Common factor
	(a) For $\zeta_0 = 0{\cdot}5$						
	Values of B_j						
	−0·0792 − 0·6082i		0·0647 − 0·1369i		0·0145 + 0·0001i		$\sqrt{(m/k)}$
	Values of $2B_j\{\psi_j^{\mathrm{y}}\} = \{\beta_j^{\mathrm{y}}\} + \mathrm{i}\{\gamma_j^{\mathrm{y}}\}$						
1	−0·1585 − 1·2163i		0·1295 − 0·2737i		0·0290 + 0·0002i		$\sqrt{(m/k)}$
2	−0·0075 − 2·1138i		0·0566 + 0·0343i		−0·0491 + 0·0234i		$\sqrt{(m/k)}$
3	0·0531 − 2·4415i		−0·1093 + 0·2485i		0·0562 − 0·0326i		$\sqrt{(m/k)}$
	Values of $\{\alpha_j^{\mathrm{y}}\}$ and $\{\beta_j^{\mathrm{y}}\}$						
1	1·1997	−0·1585	0·2875	0·1295	0·0003	0·0290	$\sqrt{(m/k)}$
2	2·1063	−0·0075	−0·0272	0·0566	−0·0243	−0·0491	$\sqrt{(m/k)}$
3	2·4378	0·0531	−0·2600	−0·1093	0·0337	0·0562	$\sqrt{(m/k)}$
	Values of $\{\alpha_j^{\mathrm{P}}\}$ and $\{\beta_j^{\mathrm{P}}\}$						
1	0·6267	−0·0828	0·4061	0·1829	0·0006	0·0556	
2	1·1004	−0·0039	−0·0384	0·0800	−0·0466	−0·0942	
3	1·2736	0·0277	−0·3673	−0·1544	0·0645	0·1077	
	Values of $\{\alpha_j^{\mathrm{A}}\}$ and $\{\beta_j^{\mathrm{A}}\}$						
1	2·2963	−0·3034	0·2035	0·0917	0·0002	0·0151	m/k
2	4·0319	−0·0143	−0·0192	0·0401	−0·0127	−0·0256	m/k
3	4·6664	0·1016	−0·1840	−0·0774	0·0176	0·0293	m/k
	(b) For $\zeta_0 = 1{\cdot}0$						
	Values of B_j						
	−0·1822 − 0·6290i		0·1751 − 0·1796i		0·0071 + 0·0138i		$\sqrt{(m/k)}$
	Values of $2B_j\{\psi_j^{\mathrm{y}}\} = \{\beta_j^{\mathrm{y}}\} + \mathrm{i}\{\gamma_j^{\mathrm{y}}\}$						
1	−0·3645 − 1·2580i		0·3502 − 0·3592i		0·0143 + 0·0277i		$\sqrt{(m/k)}$
2	−0·0441 − 2·2223i		0·1146 + 0·1362i		−0·0704 − 0·0199i		$\sqrt{(m/k)}$
3	0·0851 − 2·5718i		−0·1772 + 0·2913i		0·0921 + 0·0154i		$\sqrt{(m/k)}$
	Values of $\{\alpha_j^{\mathrm{y}}\}$ and $\{\beta_j^{\mathrm{y}}\}$						
1	1·1835	−0·3645	0·4391	0·3502	−0·0273	0·0143	$\sqrt{(m/k)}$
2	2·1866	−0·0441	−0·1010	0·1146	0·0182	−0·0704	$\sqrt{(m/k)}$
3	2·5523	0·0851	−0·3278	−0·1772	−0·0132	0·0921	$\sqrt{(m/k)}$
	Values of $\{\alpha_j^{\mathrm{P}}\}$ and $\{\beta_j^{\mathrm{P}}\}$						
1	0·6369	−0·1962	0·6113	0·4875	−0·0516	0·0270	
2	1·1768	−0·0238	−0·1407	0·1595	0·0343	−0·1329	
3	1·3736	0·0458	−0·4564	−0·2467	−0·0248	0·1738	
	Values of $\{\alpha_j^{\mathrm{A}}\}$ and $\{\beta_j^{\mathrm{A}}\}$						
1	2·1991	−0·6772	0·3154	0·2515	−0·0145	0·0076	m/k
2	4·0630	−0·0820	−0·0726	0·0823	0·0096	−0·0373	m/k
3	4·7425	0·1582	−0·2355	−0·1273	−0·0070	0·0488	m/k

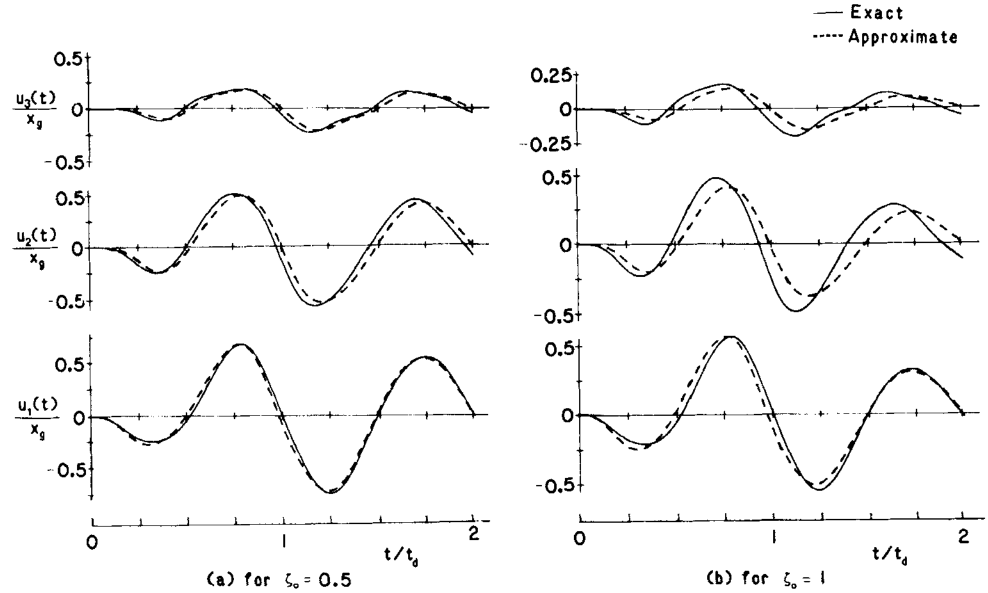

Figure 6. Interfloor deformations for system shown in Figure 1(a); system with $f_1^0 t_d = 1$ subjected to simple base motion

for a period t_0 that exceeds the duration of the excitation by the fundamental undamped natural period of the system, $T_1^0 = 1/f_1^0$; i.e. $t_0 = t_d + T_1^0$, or

$$\frac{t_0}{t_d} = 1 + \frac{1}{f_1^0 t_d}$$

Also shown in dashed lines are the corresponding approximate solutions, expressed in terms of the natural modes of vibration of the associated undamped system.

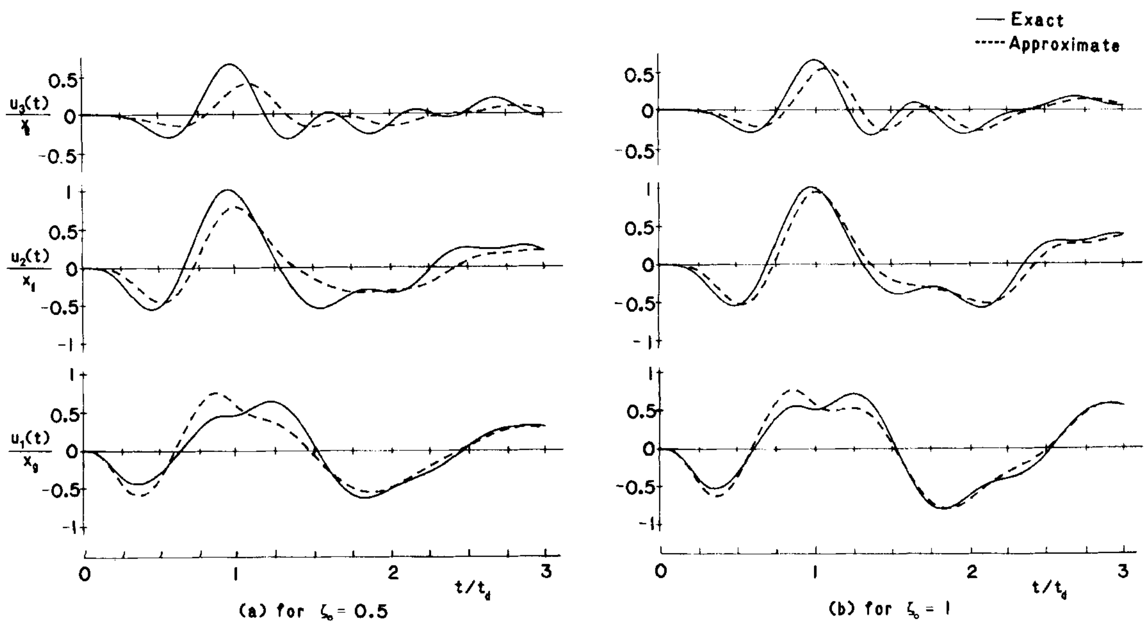

Figure 7. Interfloor deformations for system shown in Figure 1(a); system with $f_1^0 t_d = 2$ subjected to simple base motion

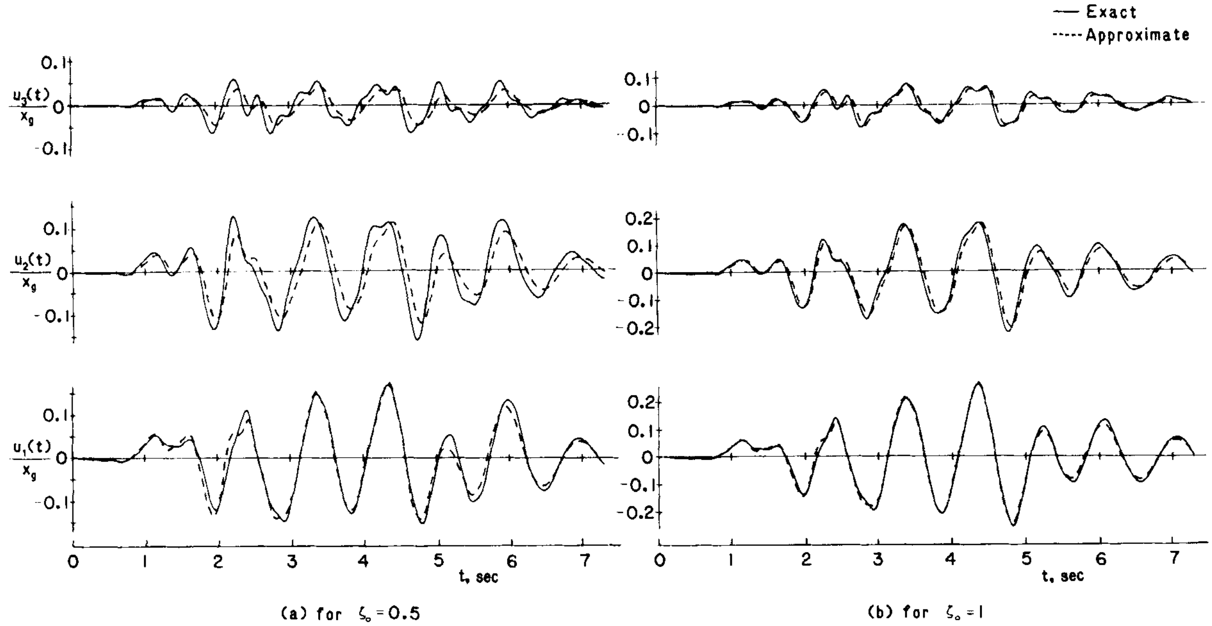

Figure 8. Interfloor deformations for system shown in Figure 1(a); system with $f_1^0 = 1$ cps subjected to El Centro earthquake record

It can be seen from Figures 6 to 8 that the differences between the approximate and exact solutions may be substantial for the damping values considered. The differences are generally larger for storey deformations than for floor displacements, particularly for the deformations of the upper parts of the structure for which the contributions of the higher modes of vibration are generally more important than for the lower parts.

These trends may better be seen in Figures 9 and 10 in which response spectra are presented for the absolute maximum floor displacements, $|(x_i)_{\max}|$, and the corresponding storey deformations, $|(u_i)_{\max}|$. These results, which are for systems with $\zeta_0 = 1$, are presented in the form of pseudovelocity values, $p_1^0|(x_i)_{\max}|$ and $p_1^0|(u_i)_{\max}|$, and they are non-dimensionalized with respect to the maximum value of the base velocity, $\dot{x}_g$. It may be recalled that p_1^0 represents the fundamental circular natural frequency of the undamped system.

Application to systems with real-valued characteristic values

The information presented so far is applicable to systems for which all characteristic values (or roots) and the associated characteristic vectors are complex-valued. The values of ζ_j in this case are less than unity, and each modal solution is given by the sum of two exponentially decaying harmonic functions. In general, there may be an even number of real-valued negative roots, each associated with a real-valued characteristic vector. The purpose of this section is to explain how these roots and vectors should be handled in a forced vibration analysis.

Let r_j and r_k be a pair of such roots, with $|r_k| > |r_j|$, and let $\{\psi_j\}$ and $\{\psi_k\}$ be the associated real-valued characteristic vectors. It is convenient to express this pair of roots in a form analogous to equation (9) as

$$r_j = -\zeta_j p_j + \tilde{p}_j \quad (68a)$$

$$r_k = -\zeta_j p_j - \tilde{p}_j \quad (68b)$$

in which

$$\tilde{p}_j = p_j \sqrt{(\zeta^2 - 1)} \quad (69)$$

and ζ_j and p_j are real, positive quantities that may be determined as follows.

On multiplying equations (68a) and (68b) and making use of equation (69), one obtains

$$p_j = \sqrt{(r_j r_k)} \quad (70)$$

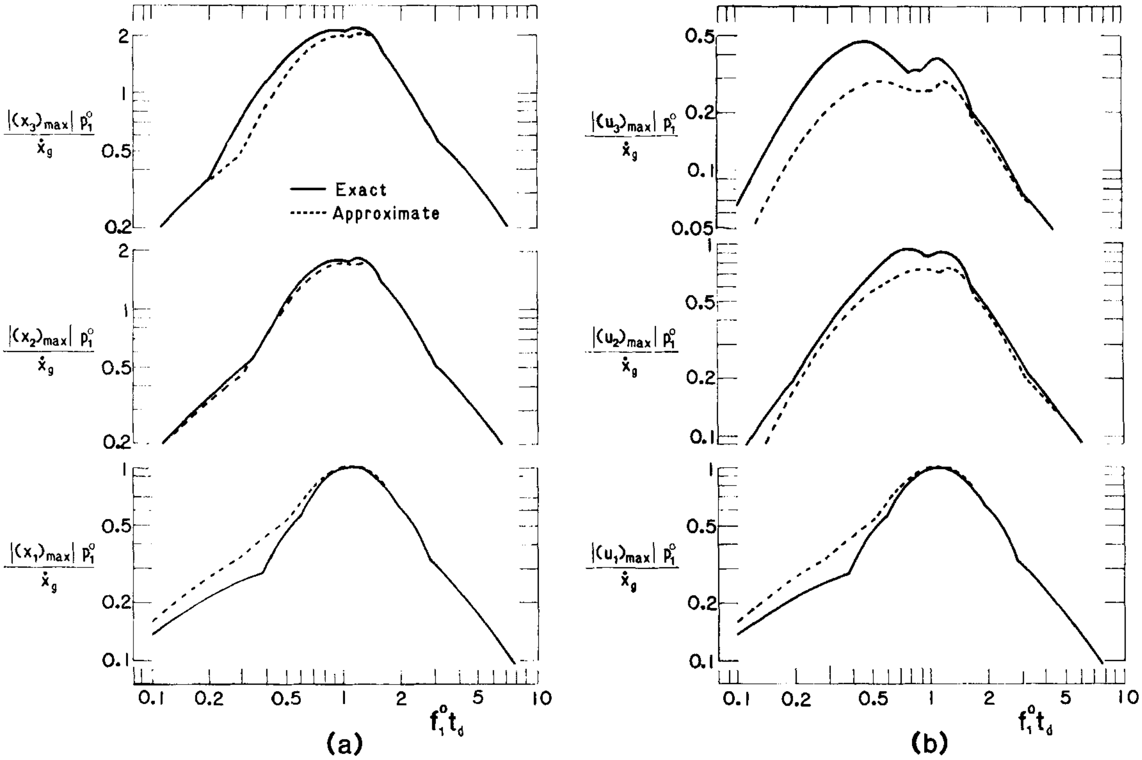

Figure 9. Response spectra for maximum floor displacements and storey deformations for system shown in Figure 1(a); system with $\zeta_0 = 1$ subjected to simple base motion

Similarly, on adding equations (68a) and (68b), one obtains

$$\zeta_j = -\frac{1}{p_j}(r_j + r_k) = -\frac{r_j + r_k}{\sqrt{(r_j r_k)}} \tag{71}$$

from which it is clear that $\zeta_j > 1$. Finally, the following expression for $\tilde{p}_j$ may be obtained by subtracting equation (68b) from equation (68a):

$$\tilde{p}_j = \frac{r_j - r_k}{2} \tag{72}$$

The modal solution corresponding to such a pair of real-valued characteristic roots is given by the sum of two exponentially decaying functions[12, 21] and hence, the resulting motion is non-oscillatory.

For a system in forced vibration, the functions $D_j(t)$ and $\dot{D}_j(t)$ in equation (57) represent the deformation and relative velocity due to the prescribed ground motion of a SDF system with the frequency and damping factor defined by equations (70) and (71), and the vector $\{\alpha_j^{\mathrm{v}}\}$ in equation (53) is given by the following modified version of equation (51):

$$\{\alpha_j^{\mathrm{v}}\} = \zeta_j\{\beta_j^{\mathrm{v}}\} - \sqrt{(\zeta_j^2 - 1)}\,\{\gamma_j^{\mathrm{v}}\} \tag{73}$$

The vectors $\{\beta_j^{\mathrm{v}}\}$ and $\{\gamma_j^{\mathrm{v}}\}$ in the latter expression and the vector $\{\beta_j^{\mathrm{v}}\}$ in equation (53) are given by

$$\{\beta_j^{\mathrm{v}}\} = B_k\{\psi_k\} + B_j\{\psi_j\} \tag{74}$$

$$\{\gamma_j^{\mathrm{v}}\} = B_k\{\psi_k\} - B_j\{\psi_j\} \tag{75}$$

in which B_j and B_k are determined from equation (43) making use of the real-valued characteristic roots, r_j and r_k, and the associated real-valued characteristic vectors, $\{\psi_j\}$ and $\{\psi_k\}$. The derivation of equations (73) to (75) is given in the last section of Appendix I.

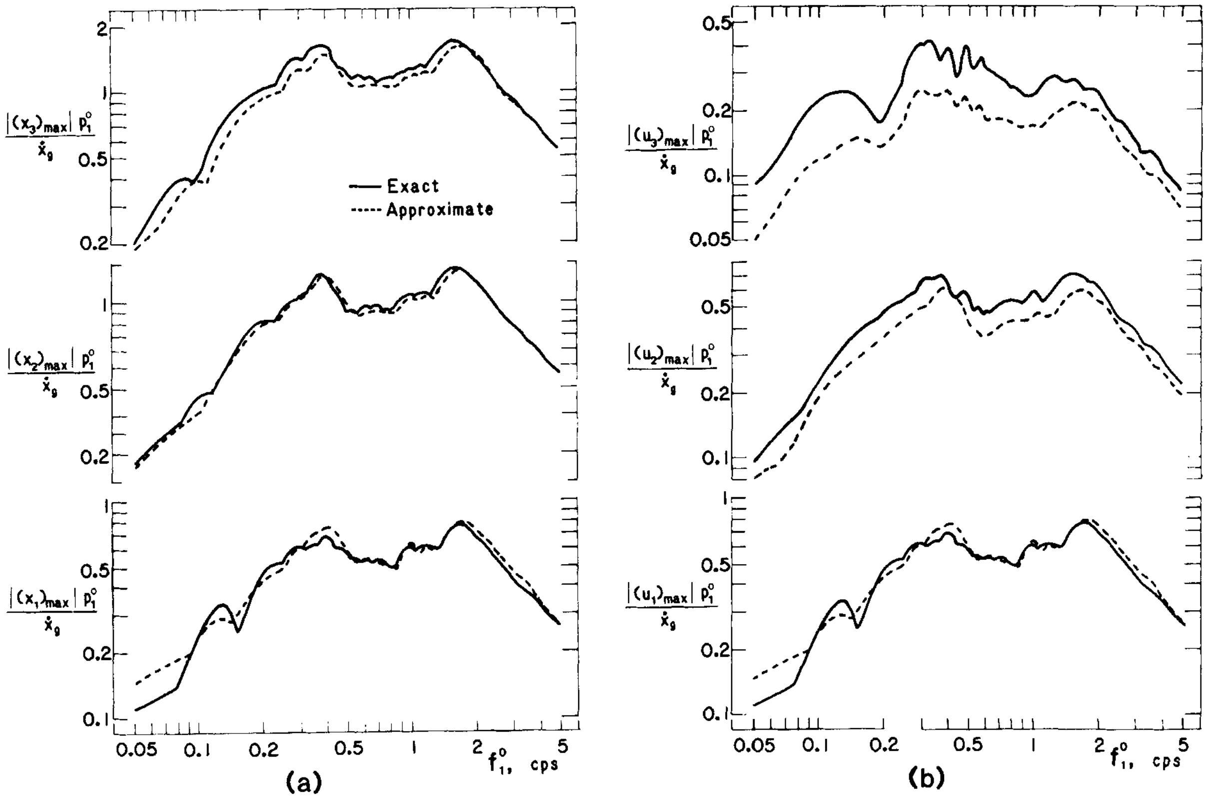

Figure 10. Response spectra for maximum floor displacements and storey deformations for system shown in Figure 1(a); system with $\zeta_0 = 1$ subjected to El Centro earthquake record

Analysis of force-excited systems

The method of analysis for the base-excited systems presented in the preceding sections can, with minor modifications, also be applied to systems excited by a set of lateral forces, $\{P(t)\}$. In the following development, these forces are considered to be of the form

$$\{P(t)\} = \{P\}g(t) \tag{76}$$

in which $\{P\}$ is an arbitrary vector with units of force and $g(t)$ is a dimensionless time function. Note that whereas the spatial distribution of the forces considered is arbitrary, their temporal variation is the same. The system is presumed to be initially at rest.

Let $\{v(\tau)\}$ be the velocity changes induced at time τ by the forces $\{P(\tau)\}$ acting in the infinitesimal time interval between τ and $\tau + \mathrm{d}\tau$. These velocities may be determined from the impulse–momentum relationship as

$$\{v(\tau)\} = [m]^{-1}\{P(\tau)\}\mathrm{d}\tau = [m]^{-1}\{P\}g(\tau)\mathrm{d}\tau \tag{77}$$

The complex-valued participation factor C_j in the expression for the resulting free vibration [equation (26)] may then be determined from equation (29) by setting $\{\dot{x}(0)\} = \{v(\tau)\}$ and $\{x(0)\} = \{0\}$. If one next introduces the quantities

$$B_j^{\mathrm{P}} = \frac{1}{p_j}\frac{\{\psi_j\}^{\mathrm{T}}\{P\}}{2r_j\{\psi_j\}^{\mathrm{T}}[m]\{\psi_j\} + \{\psi_j\}^{\mathrm{T}}[c]\{\psi_j\}} \tag{78}$$

and

$$2B_j^{\mathrm{P}}\{\psi_j\} = \{\beta_j^{\mathrm{P}}\} + \mathrm{i}\{\gamma_j^{\mathrm{P}}\} \tag{79}$$

and follows the steps taken previously in the development of the corresponding solution for base-excited systems, one obtains the following expression for the displacements:

$$\{x\} = \sum_{j=1}^{n} \left[\{\alpha_j^{\mathrm{P}}\} A_j(t) + \{\beta_j^{\mathrm{P}}\} \frac{\dot{A}_j(t)}{p_j} \right] \tag{80}$$

in which

$$\{\alpha_j^{\mathrm{P}}\} = \zeta_j \{\beta_j^{\mathrm{P}}\} = \sqrt{(1-\zeta_j^2)}\, \{\gamma_j^{\mathrm{P}}\} \tag{81}$$

$$A_j(t) = p_j^2 \int_0^t g(\tau)\, h(t-\tau)\, \mathrm{d}\tau \tag{82}$$

and $\dot{A}_j(t)$ is the time derivative of $A_j(t)$.

The quantity B_j^{P} in equation (78) and the vectors $\{\alpha_j^{\mathrm{P}}\}$, $\{\beta_j^{\mathrm{P}}\}$ and $\{\gamma_j^{\mathrm{P}}\}$ in equations (79), (80) and (81) have units of displacement, whereas the function $A_j(t)$ is dimensionless. The latter function represents the normalized displacement of a SDF system, the natural frequency and damping factor of which are the same as those of the jth mode of vibration of the prescribed multi-degree-of-freedom system, and is excited by a force of the same temporal variation as $g(t)$. The normalizing factor is the static displacement of the system induced by the peak value of the applied force. Thus

$$A_j(t) = \frac{X_j(t)}{x_{\mathrm{st}}} \tag{83}$$

in which $X_j(t)$ is the displacement of the SDF system and x_{st} is its maximum static value.

The vectors $\{\alpha_j^{\mathrm{P}}\}$ and $\{\beta_j^{\mathrm{P}}\}$ in equation (80) are the counterparts of the vectors $\{\alpha_j^{\mathrm{A}}\}$ and $\{\beta_j^{\mathrm{A}}\}$ in equation (60), and the dimensionless amplification function $A_j(t)$ is the counterpart of the normalized pseudo-acceleration function, $A_j(t)/\ddot{x}_{\mathrm{g}}$.

Application to harmonic response. The steady-state response of non-classically damped systems to a set of harmonic forces may generally be evaluated by direct solution of the governing equations of motion. However, the modal superposition method described in this paper may be preferable for systems having a large number of degrees of freedom, and its application is described briefly in this section.

For exciting forces of the form

$$\{P(t)\} = \{P\} \sin \omega t \tag{84}$$

in which ω is the circular frequency of excitation, the amplification function $A_j(t)$ and its first derivative are given by

$$A_j(t) = A_j \sin(\omega t - \theta_j) \tag{85a}$$

and

$$\dot{A}_j(t) = \omega A_j \cos(\omega t - \theta_j) \tag{85b}$$

In the latter expressions,

$$A_j = \frac{1}{\sqrt{[(1-\rho_j^2)^2 + 4\zeta_j^2 \rho_j^2]}} \tag{86a}$$

$$\rho_j = \frac{\omega}{p_j} \tag{86b}$$

and

$$\theta_j = \tan^{-1}\left(\frac{2\zeta_j \rho_j}{1-\rho_j^2}\right) \tag{86c}$$

in which $0 \leqslant \theta_j \leqslant \pi$.

Let $x_i(t)$ be the displacement of the ith floor of the system and α_{ij}^{P} and β_{ij}^{P} be the corresponding values of $\{\alpha_j^{\mathrm{P}}\}$ and $\{\beta_j^{\mathrm{P}}\}$ in equation (80). On substituting equations (85a), (85b) and (86b) into equation (80), one obtains

$$x_i(t) = \sum_{j=1}^{n} A_j [\alpha_{ij}^{\mathrm{P}} \sin(\omega t - \theta_j) + \rho_j \beta_{ij}^{\mathrm{P}} \cos(\omega t - \theta_j)] \tag{87}$$

Further, on expanding the sine and cosine functions and introducing the quantities

$$\xi_{ij} = A_j \sqrt{[(\alpha^P_{ij})^2 + (\rho_j \beta^P_{ij})^2]} \tag{88a}$$

$$\varepsilon_{ij} = \tan^{-1}\left(\frac{\alpha^P_{ij}}{\rho_j \beta^P_{ij}}\right) \tag{88b}$$

equation (87) may be rewritten as

$$x_i(t) = \sum_{j=1}^{n} \{[\xi_{ij}\sin(\theta_j + \varepsilon_{ij})]\sin \omega t + [\xi_{ij}\cos(\theta_j + \varepsilon_{ij})]\cos \omega t\} \tag{89}$$

in which ε_{ij} is understood to lie in the range 0 to 2π. The maximum value of $x_i(t)$ may finally be determined from

$$|(x_i)_{\max}| = \sqrt{\left\{\left[\sum_{j=1}^{n} \xi_{ij}\sin(\theta_j + \varepsilon_{ij})\right]^2 + \left[\sum_{j=1}^{n} \xi_{ij}\cos(\theta_j + \varepsilon_{ij})\right]^2\right\}} \tag{90}$$

Illustrative example. The steady-state response of the system shown in Figure 1(a) is evaluated for a harmonic force applied to the first floor level considering $\zeta_0 = 1$.

The natural frequencies and modes of the system and the associated damping factors are given in part (d) of Table I, and those quantities in the expressions for the response that are independent of the temporal characteristics of the forcing function are given in the upper part of Table IV. The latter quantities include the participation factors defined by equation (78); the vectors $\{\beta^P_j\}$ and $\{\gamma^P_j\}$ in equation (79); and the vectors $\{\alpha^R_j\}$ in

Table IV. Harmonic response of force-excited systems considered in example

Floor level	First mode		Second mode		Third mode		Factor or units
			Values of B^P_j				
	$-0.1561-0.2946i$		$0.0415-0.2317i$		$0.0139+0.0006i$		x_{st}
			Values of $2B_j\{\psi^P_j\} = \{\beta^P_j\}+i\{\gamma^P_j\}$				
1	$-0.3123-0.5891i$		$0.0830-0.4634i$		$0.0278+0.0011i$		x_{st}
2	$-0.2527-1.1030i$		$0.1607+0.0456i$		$-0.0465+0.0459i$		x_{st}
3	$-0.2246-1.2907i$		$0.0156+0.3196i$		$0.0527-0.0647i$		x_{st}
			Values of $\{\alpha^P_j\}$ and $\{\beta^P_j\}$				
1	0.5317	−0.3123	0.4688	0.0830	−0.0005	0.0278	x_{st}
2	1.0484	−0.2527	−0.0014	0.1607	−0.0470	−0.0465	x_{st}
3	1.2382	−0.2246	−0.3040	0.0156	0.0658	0.0527	x_{st}
			Values of ρ_j				
	1.8580		0.7183		0.5298		
			Values of A_j				
	0.3963		1.6245		1.3893		
			Values of θ_j				
	2.9039		0.6659		0.0355		rad
			Values of ε_{ij}				
1	2.3998		1.4443		6.2493		rad
2	1.9919		6.2711		4.2296		rad
3	1.8959		4.7492		1.1695		rad
			Values of ξ_{ij}				
1	0.3119		0.7677		0.0205		x_{st}
2	0.4552		0.1875		0.0737		x_{st}
3	0.5178		0.4942		0.0993		x_{st}
			Values of $\sin(\theta_j+\varepsilon_{ij})$ and $\cos(\theta_j+\varepsilon_{ij})$				
1	−0.8302	0.5574	0.8580	−0.5136	0.0016	1.0000	
2	−0.9832	0.1824	0.6082	0.7938	−0.9016	−0.4325	
3	−0.9962	0.0873	−0.7631	0.6463	0.9338	0.3577	

equation (81). The quantities that do depend on the temporal variation of the exciting force are given in the lower part of Table IV assuming that the value of the exciting frequency $\omega = \sqrt{(k/m)}$. The relevant quantities are the frequency ratios, ρ_j; the amplification factors and phase angles defined by equations (86a) and (86c); the displacement amplitudes and phase angles defined by equations (88); and the values of $\sin(\theta_j + \varepsilon_{ij})$ and $\cos(\theta_j + \varepsilon_{ij})$.

The maximum displacements of the system, $|(x_i)_{max}|$, are then determined from equation (90) to be

$$|(x_1)_{max}| = 0{\cdot}4471\, x_{st} \quad |(x_2)_{max}| = 0{\cdot}4473\, x_{st} \quad |(x_3)_{max}| = 0{\cdot}8948 x_{st}$$

in which $x_{st} = P/k$ is the static displacement of the first floor due to the peak value of the applied force.

The maximum interfloor deformations, $|(u_i)_{max}|$, are determined from equation (90) by replacing the quantities ξ_{ij} by

$$\bar{\xi}_{ij} = \xi_{ij} - \xi_{i-1,j} \tag{91}$$

The results are

$$|(u_1)_{max}| = 0{\cdot}4471 x_{st} \quad |(u_2)_{max}| = 0{\cdot}8945 x_{st} \quad |(u_3)_{max}| = 0{\cdot}4473 x_{st}$$

The maximum displacements and interfloor deformations of the system are displayed in Figure 11 over a wide range of exciting frequencies, where they are also compared with the values obtained by the approximate procedure involving the use of the undamped natural modes of vibration of the system. Note that the differences in the two sets of solutions are generally not insignificant.

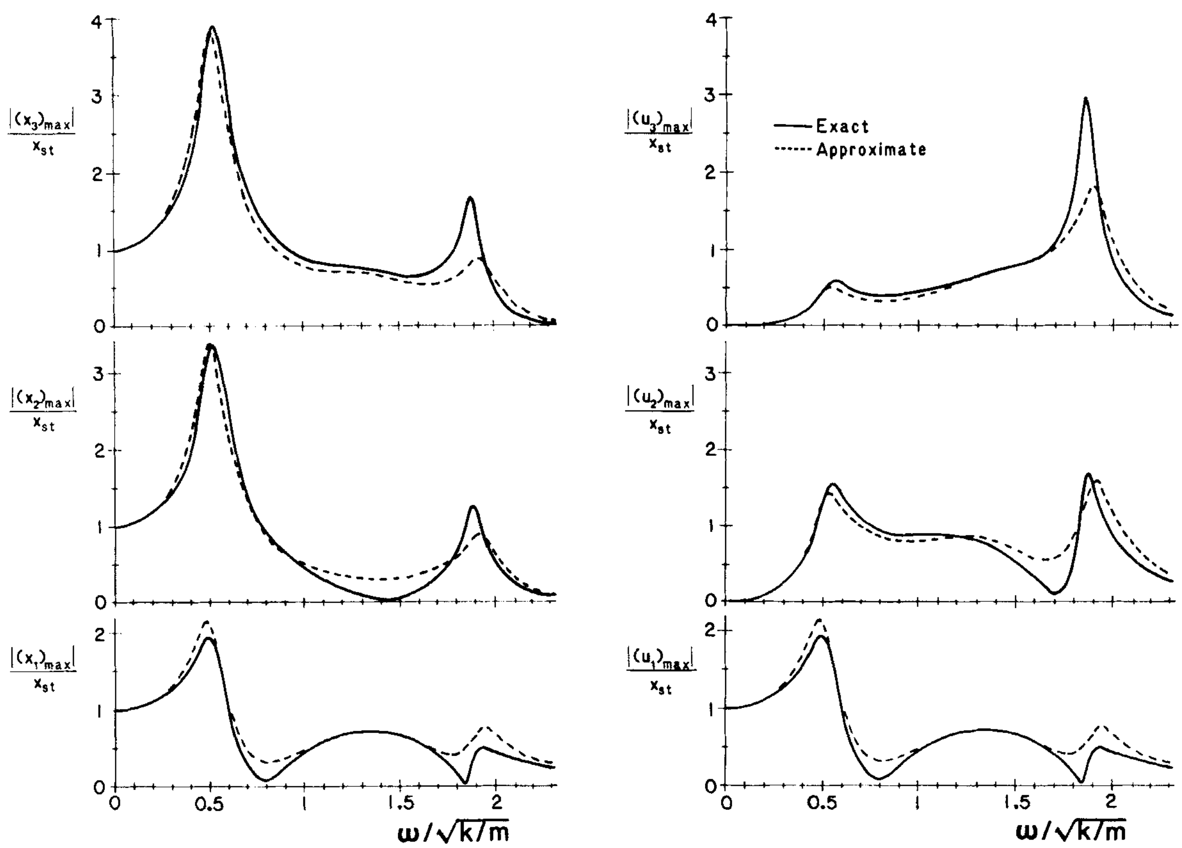

Figure 11. Response spectra for maximum floor displacements and storey deformations for system shown in Figure 1(a); system with $\zeta_0 = 1$ subjected to a harmonic force at the first floor level

CONCLUSION

With the information and the physical insight contributed in this paper, the response of a non-classically damped linear system to an arbitrary excitation may be evaluated with only minor computational effort beyond that required for the analysis of a classically damped system of the same size. The response of the system has been expressed in terms of the deformations and true relative velocities of a series of similarly excited single-degree-of-freedom systems.

Comprehensive numerical solutions have been presented for the maximum response of a three-degree-of-freedom system over a range of excitation and system parameters, and the results compared with those obtained by an approximate solution involving the use of classical modes of vibration. It has been shown that, depending on the characteristics of the excitation and of the system itself, the approximate solution may be substantially in error.

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APPENDIX I

Reduced form of equation of motion

The system of second order differential equations (1) can be reduced[3, 4] to the following first order system:

$$[A]\{\dot{z}\}+[B]\{z\} = \{Y(t)\} \tag{A1}$$

in which $[A]$ and $[B]$ are matrices of size $2n$ by $2n$ given by

$$[A] = \begin{bmatrix} [0] & [m] \\ [m] & [c] \end{bmatrix} \qquad [B] = \begin{bmatrix} -[m] & [0] \\ [0] & [k] \end{bmatrix} \tag{A2}$$

and $\{z\}$ and $\{Y(t)\}$ are vectors of $2n$ elements given by

$$\{z\} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} \qquad \{Y(t)\} = \begin{Bmatrix} \{0\} \\ -[m]\{1\}\ddot{x}_g(t) \end{Bmatrix} \tag{A3}$$

The solution of the homogeneous form of equation (A1) may be taken as

$$\{z\} = \{Z\}\mathrm{e}^{rt} \tag{A4}$$

where r is a characteristic value and $\{Z\}$ is the associated characteristic vector of $2n$ elements. The lower n elements of $\{Z\}$ represent the desired modal displacements, $\{\psi\}$, and the upper n elements represent the corresponding modal velocities, $r\{\psi\}$; i.e.

$$\{Z\} = \begin{Bmatrix} r\{\psi\} \\ \{\psi\} \end{Bmatrix} \tag{A5}$$

Substituting equation (A4) into the homogeneous form of equation (A1), one obtains the characteristic value problem defined by equation (4).

Form of characteristic values

Substitution of $\{x\} = \{\psi_j\}\mathrm{e}^{r_j t}$ into the homogeneous form of equation (1) leads to

$$[m]\{\psi_j\}r_j^2+[c]\{\psi_j\}r_j+[k]\{\psi_j\} = \{0\} \tag{A6}$$

and premultiplication by the transpose of the complex conjugate of $\{\psi_j\}$ leads to

$$\{\bar{\psi}_j\}^{\mathrm{T}}[m]\{\psi_j\}r_j^2+\{\bar{\psi}_j\}^{\mathrm{T}}[c]\{\psi_j\}r_j+\{\bar{\psi}_j\}^{\mathrm{T}}[k]\{\psi_j\} = 0 \tag{A7}$$

where the superscript T denotes a transposed vector. Each of the three matricial products represents a positive real number. On letting

$$m_j^* = \{\bar{\psi}_j\}^{\mathrm{T}}[m]\{\psi_j\} \tag{A8a}$$

$$c_j^* = \{\bar{\psi}_j\}^{\mathrm{T}}[c]\{\psi_j\} \tag{A8b}$$

$$k_j^* = \{\bar{\psi}_j\}^{\mathrm{T}}[k]\{\psi_j\} \tag{A8c}$$

equation (A7) can be written as

$$m_j^* r_j^2 + c_j^* r_j + k_j^* = 0 \tag{A9}$$

which is recognized to be the characteristic or frequency equation for a single-degree-of-freedom system with mass m_j^*, damping coefficient c_j^* and stiffness k_j^*. Proceeding in the usual manner and letting

$$p_j = \sqrt{(k_j^*/m_j^*)} \tag{A10}$$

and

$$2\zeta_j p_j = c_j^*/m_j^* \tag{A11}$$

equation (A9) can be rewritten as

$$r_j^2 + 2\zeta p_j r_j + p_j^2 = 0 \tag{A12}$$

the roots of which are

$$r_j = -\zeta_j p_j + \mathrm{i}\tilde{p}_j \tag{A13a}$$

and

$$\bar{r}_j = -\zeta_j p_j - \mathrm{i}\tilde{p}_j \tag{A13b}$$

where

$$\tilde{p}_j = p_j \sqrt{(1-\zeta_j^2)} \tag{A13c}$$

Similar derivations have been given previously.[5, 9, 17, 21]

Orthogonality of modes

Since $[A]$ and $[B]$ in equation (4) are real symmetric matrices, the characteristics vectors $\{Z_j\}$ and $\{Z_k\}$ corresponding to any pair of distinct characteristic values r_j and r_k satisfy the orthogonality relations

$$\{Z_j\}^{\mathrm{T}}[A]\{Z_k\} = 0 \tag{A14}$$

and

$$\{Z_j\}^{\mathrm{T}}[B]\{Z_k\} = 0 \tag{A15}$$

These relations also hold true for a complex conjugate pair of vectors $\{Z_j\}$ and $\{\bar{Z}_j\}$ since the associated characteristic values, r_j and $\bar{r}_j$, are different.

On making use of equations (A2) and (A5), equation (A14) reduces to equation (12), and equation (A15) reduces to equation (13).

Damping matricial product for classically damped systems

For classically damped systems, $r_j = -\zeta_j p_j^0 + \mathrm{i}\tilde{p}_j^0$, $\{\psi_j\} = \{\phi_j\}$ and

$$[k]\{\phi_j\} = (p_j^0)^2[m]\{\phi_j\} \tag{A16}$$

On making use of these facts, equation (A6) reduces to

$$[c]\{\phi_j\} = 2\zeta_j p_j^0[m]\{\phi_j\} \tag{A17}$$

Finally, on multiplying both sides of equation (A17) by the transpose of $\{\psi_k\}$ and making use of the symmetry of $[m]$ and $[c]$ and of the fact that $\{\psi_k\} = \{\phi_k\}$, one obtains

$$\{\psi_j\}^{\mathrm{T}}[c]\{\psi_k\} = \{\phi_j\}^{\mathrm{T}}[c]\{\phi_k\} = 2\zeta_j p_j^0 \{\phi_j\}^{\mathrm{T}}[m]\{\phi_k\} \tag{A18}$$

A similar relationship holds for an arbitrary real-valued vector $\{x(0)\}$; i.e.

$$\{\psi_j\}^{\mathrm{T}}[c]\{x(0)\} = 2\zeta_j p_j^0 \{\phi_j\}^{\mathrm{T}}[m]\{x(0)\} \tag{A19}$$

Complex-valued participation factors for free vibration

The complete solution of the homogeneous form of equation (A1) is given by

$$\{z\} = \sum_{j=1}^{n} C_j\{Z_j\}e^{r_j t} + \sum_{j=1}^{n} \bar{C}_j\{\bar{Z}_j\}e^{\bar{r}_j t} \tag{A20}$$

in which the participation factors, C_j and $\bar{C}_j$, may be determined from the initial conditions of the problem as follows. Let

$$\{z(0)\} = \begin{Bmatrix} \{\dot{x}(0)\} \\ \{x(0)\} \end{Bmatrix} \tag{A21}$$

be the vector of the velocities and displacements of the system at $t = 0$. Then

$$\{z(0)\} = \sum_{j=1}^{n} C_j\{Z_j\} + \sum_{j=1}^{n} \bar{C}_j\{\bar{Z}_j\} \tag{A22}$$

On premultiplying both sides of this equation by $\{Z_k\}^T[A]$ and making use of the orthogonality condition defined by equation (A14), it can be shown that all terms on the right-hand member of the resulting expression, except for the $k = j$ term, vanish. This leads to

$$C_j = \frac{\{Z_j\}^T[A]\{z(0)\}}{\{Z_j\}^T[A]\{Z_j\}} \tag{A23}$$

which, on making use of equations (A2), (A5) and (A21) reduces to equation (29).

Initial conditions that excite a single mode

When expressed in terms of the vectors $\{z\}$ and $\{Z_k\}$, equations (35a) and (35b) may be written as

$$\{z(0)\} = 2\,\mathrm{Re}[C_k\{Z_k\}] = C_k\{Z_k\} + \bar{C}_k\{\bar{Z}_k\} \tag{A24}$$

The participation factors C_j may then be determined from equation (A23) as

$$C_j = \frac{\{Z_j\}^T[A](C_k\{Z_k\} + \bar{C}_k\{\bar{Z}_k\})}{\{Z_j\}^T[A]\{Z_j\}} \tag{A25}$$

Because of the orthogonality condition defined by equation (A14) this expression vanishes, except when $j = k$, in which case $C_j = C_k$.

Systems with real-valued roots

The motion represented by a linear combination of two real-valued characteristic vectors, $\{\psi_j\}$ and $\{\psi_k\}$, and the associated characteristic values, r_j and r_k, is given by

$$\{x\} = C_j\{\psi_j\}e^{r_j t} + C_k\{\psi_k\}e^{r_k t} \tag{A26}$$

in which the quantities C_j and C_k in this case are real-valued constants that can be evaluated from equation (29). On making use of equations (68) and of the relationship between exponential and hyperbolic functions, equation (A26) may be rewritten as

$$\{x\} = e^{-\zeta_j p_j t}[\{\beta_j\}\cosh \tilde{p}_j t - \{\gamma_j\}\sinh \tilde{p}_j t] \tag{A27}$$

where

$$\{\beta_j\} = C_k\{\psi_k\} + C_j\{\psi_j\} \tag{A28}$$

$$\{\gamma_j\} = C_k\{\psi_k\} - C_j\{\psi_j\} \tag{A29}$$

For a system subjected to a set of unit initial velocity changes, $\{\dot{x}(0)\} = \{1\}$, the participation factors C_j and C_k must be replaced by B_j and B_k, and the vectors $\{\beta_j\}$ and $\{\gamma_j\}$ must be replaced by $\{\beta_j^v\}$ and $\{\gamma_j^v\}$, respectively. Equations (A28) and (A29) then reduce to equations (74) and (75). Further, on recalling that the impulse response function for an overdamped single-degree-of-freedom system is given by

$$h_j(t) = \frac{1}{\tilde{p}_j} e^{-\zeta_j p_j t} \sinh \tilde{p}_j t \tag{A30}$$

it can be shown that equation (A26) leads to

$$\{x\} = \{\alpha_j^{\mathrm{v}}\} p_j h_j(t) + \{\beta_j^{\mathrm{v}}\} \dot{h}_j(t) \tag{A31}$$

in which $\tilde{p}_j$ is defined by equation (69) and $\{\alpha_j^{\mathrm{v}}\}$ is defined by equation (73).

APPENDIX II

Notation

$A_j(t)$	pseudo-acceleration of a SDF system
B_j	participation factor for base-excited system
B_j^{P}	participation factor for force-excited system
c	damping coefficient
$[c]$	damping matrix of system
C_j	participation factor for system in free vibration
$D_j(t)$	deformation of a base-excited SDF system
$\dot{D}_j(\mathrm{t})$	relative velocity of a base-excited SDF system
$h_j(t)$	impulse response function for a SDF system
i	$\sqrt{-1}$; when used as a subscript, it indicates level of floor or storey
j	integer number indicating order of mode under consideration
k	stiffness coefficient; when used as a subscript, it indicates order of mode
$[k]$	stiffness matrix of system
m	mass coefficient
$[m]$	mass matrix of system
n	number of degrees of freedom in a system
p_j, $\tilde{p}_j$	pseudo-undamped and damped circular frequency of jth mode, respectively
p_j^0	$2\pi f_j^0$ = circular natural frequency of jth mode of associated undamped system
r	characteristic value
SDF	single-degree-of-freedom system
T_1^0	natural period of fundamental mode of associated undamped system
t	time
t_{d}	duration of excitation
t_0	$t_{\mathrm{d}} + T_1^0$
$\{u\}$	vector of interfloor deformations
u_i	interfloor deformation of ith storey
$\lvert(u_i)_{\max}\rvert$	absolute maximum value of u_i
$V_j(t)$	pseudovelocity of a SDF system
$\{x\}$	vector of displacements relative to moving base for a base-excited system, and of absolute displacements for a force-excited system
$\{x(0)\}$, $\{\dot{x}(0)\}$	initial displacements and initial velocities of system, respectively
x_i, $x_i(t)$	displacement of ith floor or level
$\lvert(x_i)_{\max}\rvert$	absolute maximum value of x_i
$\ddot{x}_{\mathrm{g}}(t)$	acceleration of the moving base
$\ddot{x}_{\mathrm{g}}$	peak value of $\ddot{x}_{\mathrm{g}}(t)$
$\{Z\}$	characteristic vector of size $2n$
$\mathcal{A}_j(t)$	amplification function in a force-excited system
$\{\alpha_j^{\mathrm{v}}\}$	modal configuration defined by equation (51)

$\{\alpha_j^P\}$	modal configuration defined by equation (81)
$\{\beta_j\}, \{\gamma_j\}$	modal configurations defined by equation (22)
$\{\beta_j^v\}, \{\gamma_j^v\}$	modal configurations defined by equation (45)
$\{\beta_j^P\}, \{\gamma_j^P\}$	modal configurations defined by equation (79)
ζ_j	modal damping factor
ζ_0	dimensionless damping coefficient in equation (17)
$\{\phi_j\}, \{\chi_j\}$	real part and imaginary part of jth complex-valued natural mode, $\{\psi_j\}$, respectively
$\{\psi\}$	characteristic vector or natural mode of system

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