DYNAMIC *K*-GRAPHS: AN ALGORITHM FOR DYNAMIC GRAPH LEARNING AND TEMPORAL GRAPH SIGNAL CLUSTERING

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Outline

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GSP Background

Dynamic *K*-graphs Algorithm

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Introduction

- Graph signal processing (GSP) is a tool for representing and inferring the data.
- Various applications of GSP in many areas
- Necessary to know the underlying graph in many of the applications
- Introducing of many graph learning algorithm

Introduction

- Graph learning algorithms:
  - Single graph learning algorithms:
    - Graphical LASSO [Friedman et al., 2008]
    - Estimate Laplacian matrix with the smoothness assumption of graph signals [Kalofolias, 2016]
  - Multiple graph learning algorithms:
    - Gaussian mixture model (GMM)-based multiple graph learning **(GLMM)** [Maretic et al., 2020]
    - K-means-based multiple graph learning **(K-graphs)** [Araghi et al., 2019]

- [Friedman et al., 2008] J. Friedman, T. Hastie, and R. Tibshirani, "Sparse inverse covariance estimation with the graphical lasso," Biostatistics, 2008.
- [Kalofolias, 2016] V. Kalofolias, "How to learn a graph from smooth signals," AISTATS, 2016.
- [Maretic et al., 2020] H. P. Maretic and P. Frossard, "Graph Laplacian Mixture Model," TSIPN, 2020.
- [Araghi et al., 2019] H. Araghi, M. Sabbaqi, and M. Babaie-Zadeh, "K-Graphs: An algorithm for graph signal clustering and multiple graph learning," IEEE Signal Process. Lett., 2019.

Introduction

- Graph learning algorithms *(continued)*:
  - Dynamic graph learning algorithms:
    - Dynamic graphical LASSO [Hallac et al., 2017a]
    - Time-varying graph learning based on smoothness assumption [Kalofolias et al., 2017]
    - Dynamic graphical LASSO with clustering capability **(TICC)** [Hallac et al., 2017b]

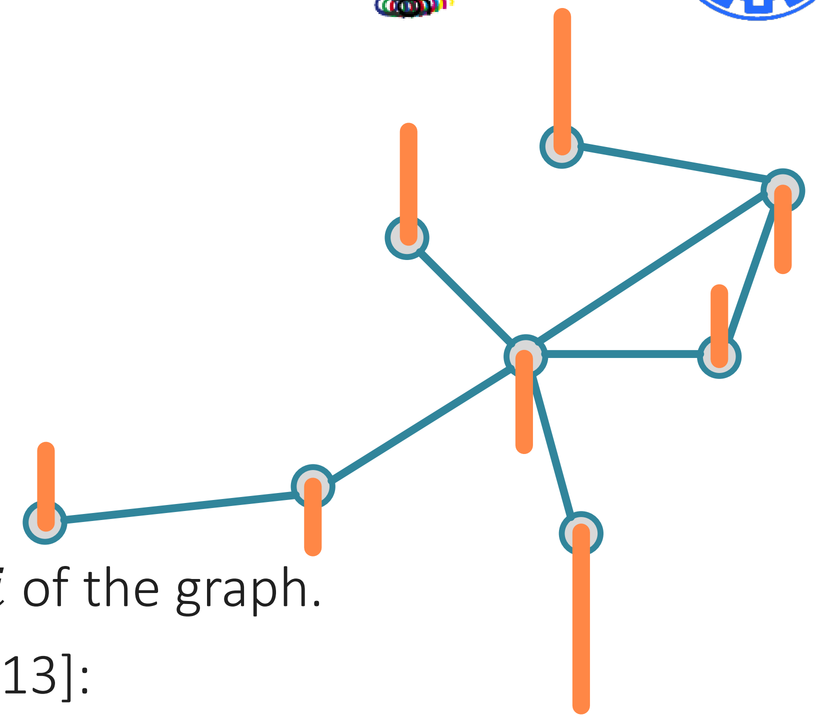
- [Hallac et al., 2017a] D. Hallac, Y. Park, S. Boyd, and J. Leskovec, "Network inference via the time-varying graphical lasso," SIGKDD, 2017.
- [Kalofolias et al., 2017] V. Kalofolias, A. Loukas, D. Thanou, and P. Frossard, "Learning time varying graphs," ICASSP, 2017.
- [Hallac et al., 2017b] D. Hallac, S. Vare, S. Boyd, and J. Leskovec, "Toeplitz inverse covariance-based clustering of multivariate time series data," SIGKDD, 2017.

Introduction

- Main contributions:
  - Dynamic K-graph
  - Extracting the change points and time intervals
  - Clustering the temporal graph signals (multivariate time series)
  - More interpretable graph structures
  - Higher clustering accuracy in simulation results

GSP Background

- A weighted and undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathbf{A})$
  - $\mathcal{V}$ is the set of nodes.
  - $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is the set of edges.
  - $\mathbf{A}$ is the symmetric weighted adjacency matrix.
  - $\mathbf{D}$ is the diagonal degree matrix.
  - $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the graph Laplacian matrix.
- Graph signal $\mathbf{x}$
  - A vector whose i-th entry assigns a real value to the node i of the graph.
- Smoothness of graph signal $\mathbf{x}$ over a graph [Shuman et al., 2013]:



$$\mathbf{x}^T \mathbf{L} \mathbf{x} = \sum_{i,j} \mathbf{A}[i,j] \left(\mathbf{x}[i] - \mathbf{x}[j]\right)^2$$

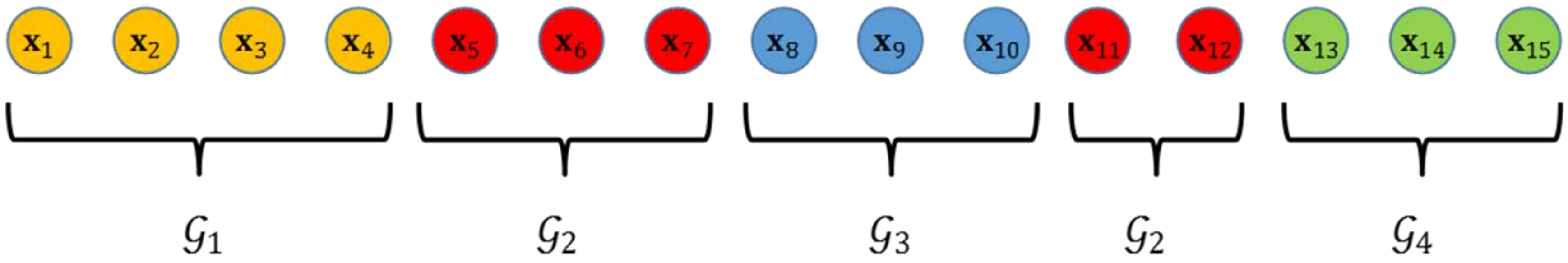
[Shuman et al., 2013] D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, "The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains," IEEE Signal Process. Mag., 2013.

Dynamic K-graphs Algorithm

- Problem definition:
  - Multivariate time-series $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_T]$ consisting of T temporal graph signals
  - Divided into different time intervals
  - Signals in each time interval coming from one undirected graphs
  - Graph of each time interval is chosen from K unknown graphs $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_K$

- Goal:
  - Jointly estimating the time intervals and the set of graphs $\{\mathcal{G}_k\}_{k=1}^{K}$

- Example:



Dynamic K-graphs Algorithm

- Problem formulation:

$$\min_{\substack{\{\mathbf{L}_k\}_{k=1}^K, \\ \{n_t\}_{t=1}^T}} \sum_{k=1}^{K} \sum_{\mathbf{x}\in\mathcal{X}_k} \mathbf{x}^T \mathbf{L}_k \mathbf{x} + \sum_{k=1}^{K} f(\mathbf{L}_k) + \alpha \sum_{t=2}^{T} \mathbb{1}(n_t \neq n_{t-1}),$$

Smoothness of graph signals on graph $\mathcal{G}_k$

Regularization terms for Laplacian matrices

Temporal consistency

$$\begin{aligned} \text{s.t.} \quad & \mathbf{L}_k \in \mathcal{L},\ (1 \le k \le K), \\ & n_t \in \{1,\ldots,K\},\ (1 \le t \le T), \\ & \mathcal{X}_k = \{\mathbf{x}_t : t \in \{1,\ldots,T\}, n_t = k\},\ (1 \le k \le K) \end{aligned}$$

Dynamic K-graphs Algorithm

- Solving the problem with alternating minimization for $\{\mathbf{L}_k\}_{k=1}^{K}$ and $\{n_t\}_{t=1}^{T}$:
  - Initialization step: each graph signal is randomly and independently assigned to one of K clusters.
  - First Step: fixing n_t's, the optimization problem is solved for $\mathbf{L}_k$.

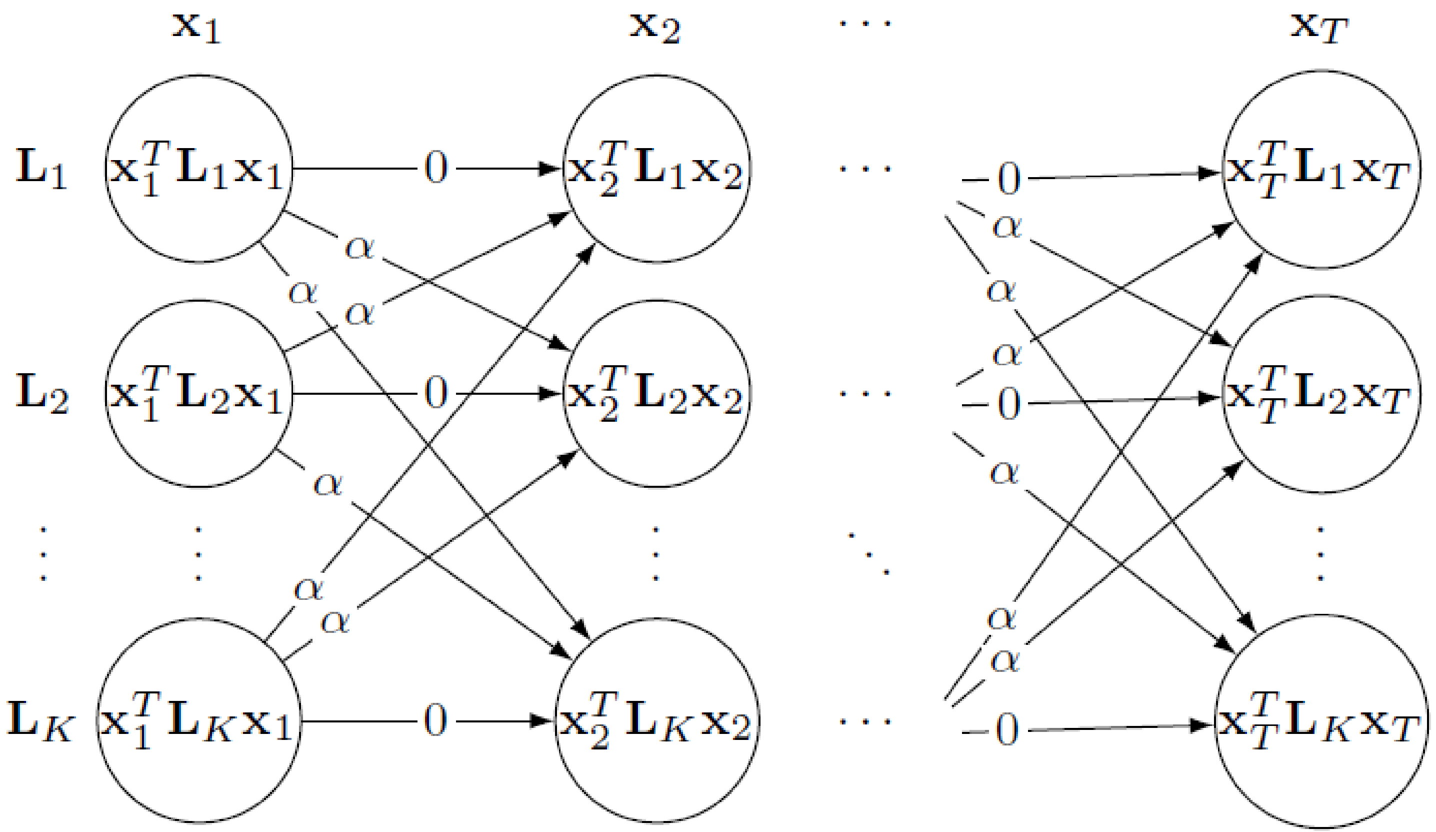
$$\mathbf{L}_k = \underset{\mathbf{L}\in\mathcal{L}}{\operatorname{argmin}} \sum_{\mathbf{x}\in\mathcal{X}_k} \mathbf{x}^T\mathbf{L}\mathbf{x} + f(\mathbf{L})$$

  - Second step: fixing Laplacian matrices $\mathbf{L}_k$'s, the graph signals are assigned to one of K clusters.

$$\begin{aligned} \min_{\{n_t\}_{t=1}^{T}} \quad & \sum_{k=1}^{K}\sum_{\mathbf{x}\in\mathcal{X}_k} \mathbf{x}^T\mathbf{L}_k\mathbf{x} + \alpha\sum_{t=2}^{T}\mathbb{1}(n_t \neq n_{t-1}), \\ \text{s.t.} \quad & n_t \in \{1,\dots,K\}, \quad (1\le t\le T), \\ & \mathcal{X}_k = \{\mathbf{x}_t : t\in\{1,\dots,T\}, n_t = k\}, \quad (1\le k\le K) \end{aligned}$$

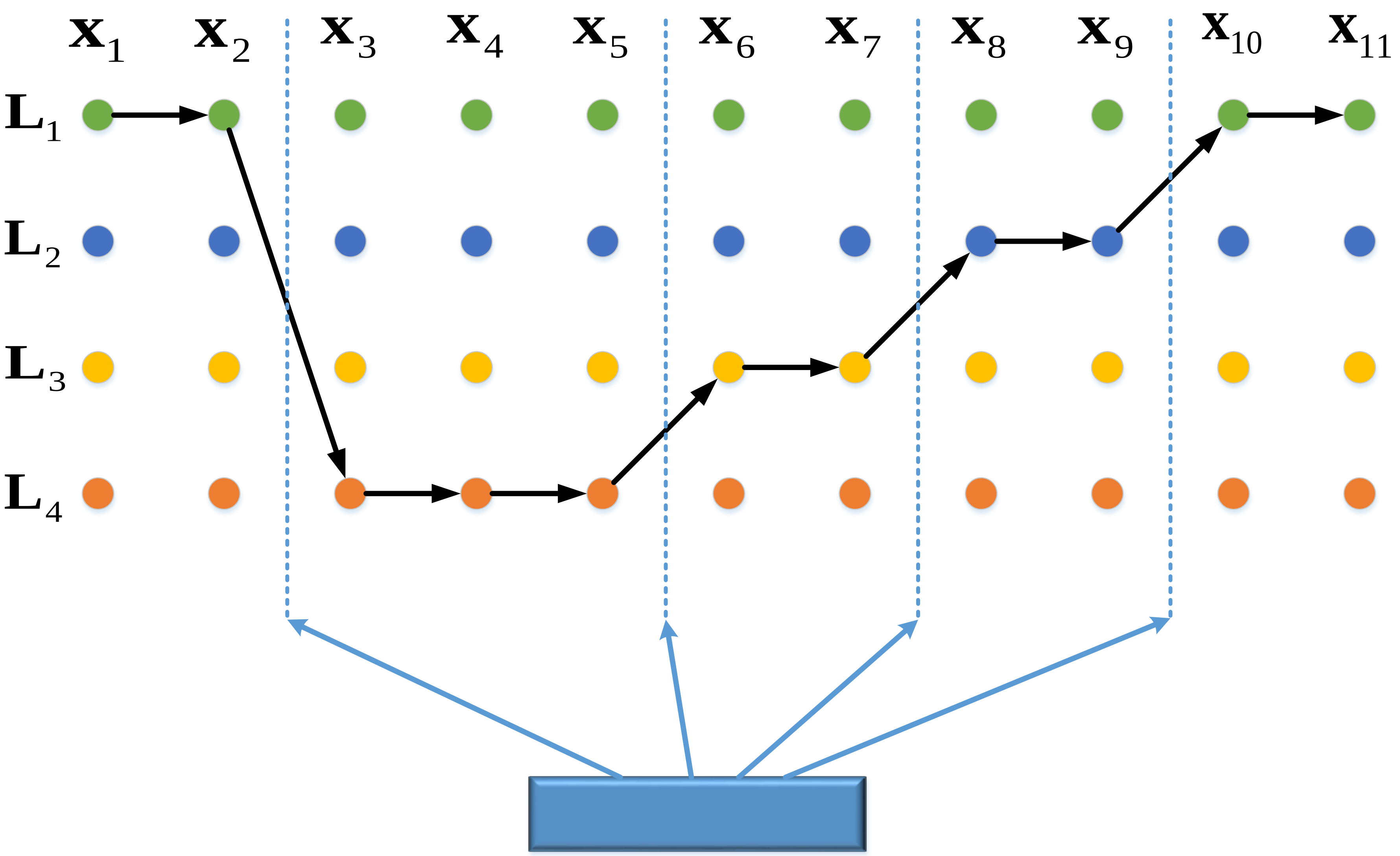
Dynamic K-graphs Algorithm

- Using dynamic programming method, such as Viterbi algorithm [Viterbi, 1967], to solve the minimization problem in the second step



[Viterbi, 1967] A. Viterbi, "Error bounds for convolutional codes and an asymptotically optimum decoding algorithm," IEEE Trans. Inf. Theory, 1967.

Dynamic K-graphs Algorithm



Simulation Results

- Dynamic K-graphs performance in clustering accuracy and graph learning
- Comparing with other algorithms:

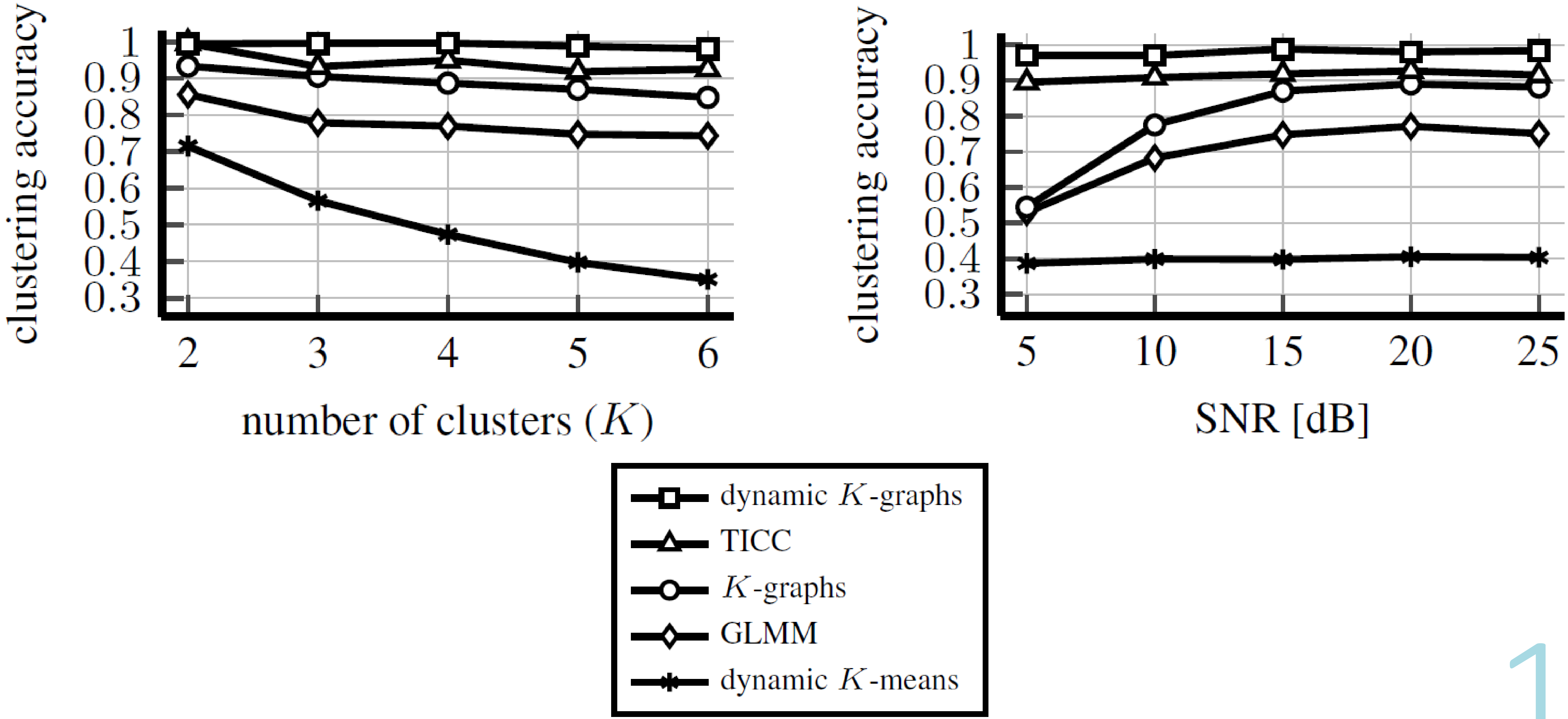
  TICC   K-graphs   GLMM   Dynamic K-means

- The time series $\mathbf{X}$ consist of $T = 1000$ temporal graph signals $\mathbf{x}_t \in \mathbb{R}^{30}$ which are corrupted by additive white noise.
- The number of change points is randomly chosen from the integer set $\{K+1, \dots, 3K\}$.
- The signals in each time interval are smooth over one graph.
- There are K graphs $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_K$.
- The similarity of estimated Laplacian matrices to the true ones is measured by the following SNR

$$\mathrm{SNR}_{\mathbf{L}} = 10 \log_{10} \left(\frac{\sum_{k=1}^{K} \|\mathbf{L}_k^{(\mathrm{true})}\|_F^2}{\sum_{k=1}^{K} \|\mathbf{L}_k^{(\mathrm{true})} - \mathbf{L}_k\|_F^2} \right)$$

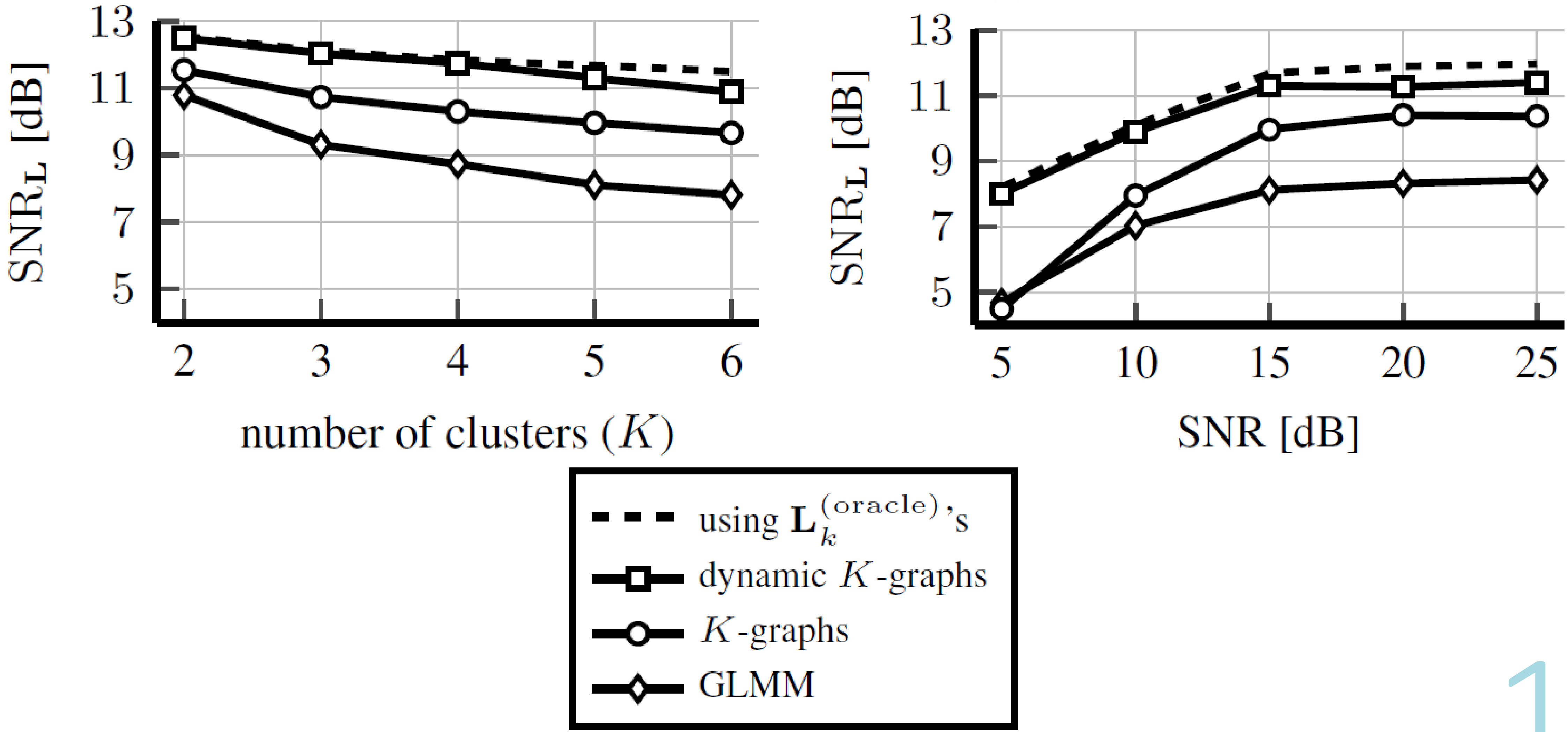
Experiment 1

- Clustering Performance Evaluation:



Experiment 2

- Graph Learning Performance Evaluation:



Conclusions

- Dynamic *K*-graphs, a dynamic graph learning algorithm
- Segmenting the time series into different time interval
- Capable of temporal graph signal clustering
- High clustering accuracy and good graph learning performance in numerical simulations

Thank You!