INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS D: APPLIED PHYSICS
J. Phys. D: Appl. Phys. **36** (2003) 1644–1650 PII: S0022-3727(03)58851-9

The effective conductivity of three-phase composite materials with circular cylindrical inclusions

A Moosavi and P Sarkomaa

Department of Energy and Environmental Technology, Lappeenranta University of Technology, PO Box 20, Fin-53851, Lappeenranta, Finland

E-mail: Ali.Moosavi@lut.fi

Received 28 January 2003
Published 18 June 2003
Online at stacks.iop.org/JPhysD/36/1644

Abstract
We extend the Rayleigh method for the calculation of the effective conductivity to three-phase composite materials. The materials under study consist of two types of circular cylinders in a periodic arrangement embedded in a matrix. Highly accurate values for lattice sums were obtained using algorithms which have been recently developed. A series of explicit formulations, which are used to facilitate the calculation of the effective conductivity of the composites under study, are reported. We also perform a series of numerical calculations to study the behaviour of these composites.

1. Introduction

Multi-phase heterogeneous systems can be found in a wide range of practical processes and are of considerable technological importance. Colloidal dispersions, emulsions, solid rocket propellant, oil-filled porous rocks, concrete, reinforced materials are a few examples of such systems. To handle these systems optimally one needs to know how they behave at the macroscopic level; therefore a great deal of interest has been focused on relating the microstructural and macroscopic properties of these systems, such as the effective thermal, or electrical, conductivity [1, 2], dielectric permittivity [3, 4] and permeability of a porous medium [5].

One common type of these systems consists of a matrix and a series of dispersed phases. Of these systems those with circular cylinders were among the first to be studied by researchers for deriving the effective conductivity, as is the case in Rayleigh's studies [1]. Rayleigh took into account a rectangular array of circular cylinders and showed that to completely relate the microstructural and macroscopic properties, the effect of interaction between the cylinders should be taken into account. This fact was later extensively used for obtaining more accurate analytical relations for calculating the effective conductivity of these composite systems [6, 7]. Recently, some attempts have been made to generalize the study by developing formulations for multi-phase cases [8]. Clearly, this is an important task since the behaviour demonstrated by multi-phase systems can be completely different from that understood on the basis of two-phase systems and further investigation is necessary.

In this paper, we are concerned with the problem of calculating the effective conductivity of three-phase periodic structures composed of two types of circular cylinders. The unit cell of the composites under study is a rectangular cylinder with a circular cylinder of type one at the centre and a circular cylinder of type two at each corner, as depicted in figure 1. The phases can be solid or stagnant fluid. The

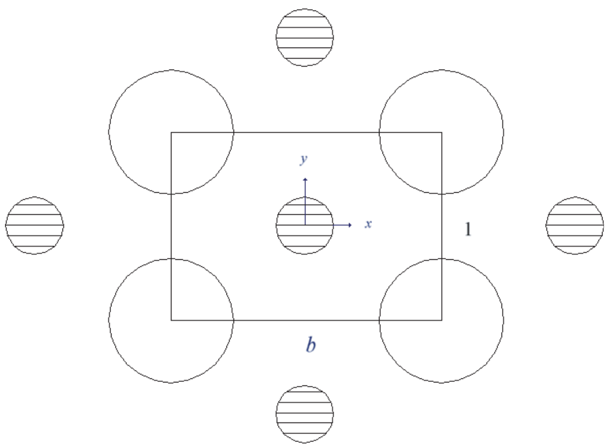

Figure 1. The unit cell of the three-phase composites under study. Cylinders of the same type have a distance equal to a unity in the x-direction and b in the y-direction.

configuration of the geometry selected for study makes it possible to construct many periodic structures, which can be widely found in literature. In what follows, we first develop the Rayleigh method to the three-phase composites. Then, we verify the algorithm, comparing our results with some existing results. Finally, we inspect the behaviour of the systems. We shall use terms and notations appropriate to the case of thermal conduction for convenience. It is worth noting that the results of this study can be applied to many transport properties besides thermal or electrical conductivity [9, 10].

2. Governing equations

Suppose that the origin of cartesian coordinates has been placed at the centre of a cylinder of type one in a unit cell of the system in which the x- and y-axes are parallel to the sides of the unit cell (see figure 1). Furthermore, assume that the matrix of the composite under study has unit conductivity and the periodicity of the system in the x-direction is equal to b and in the y-direction is equal to a unity for greater generality. Applying these conditions, we denote the conductivity of the cylinders by k_1 and k_2, the radii a_1 and a_2 and the volume fractions f_1 and f_2 for the cylinders of type one and two, respectively.

For the case $b \neq 1$, the effective conductivity of the system in the x-direction (parallel) would be different from that of the y-direction (perpendicular) and one should calculate both these conductivities. In order to simplify the presentation without loss of generality herein, we only consider the parallel direction. The necessary comments for the perpendicular direction shall be outlined in a separate section.

Let us apply a uniform temperature gradient of unit magnitude externally to the system along the x-axis in the negative direction. By taking the centre of any cylinder of type i ($i = 1$ or 2) as the origin of polar coordinates (r, θ), the temperature within that cylinder and outside it through the matrix can be given as

$$T_i(r,\theta) = C_{0,i} + \sum_{n=1}^{\infty} C_{2n-1,i} r^{2n-1} \cos(2n-1)\theta, \quad (1)$$

$$T_{m,i}(r,\theta) = A_{0,i} + \sum_{n=1}^{\infty} (A_{2n-1,i} r^{2n-1} + B_{2n-1,i} r^{-2n+1}) \times \cos(2n-1)\theta, \quad (2)$$

where the set of coefficients $C_{2n-1,i}$, $A_{2n-1,i}$ and $B_{2n-1,i}$ are unknowns to be determined. The periodicity of the system implies that these coefficients be exactly the same for all cylinders of the same type. $C_{0,i}$ and $A_{0,i}$ express the average of the temperature within the cylinder and outside it, respectively. Thus, they are exactly the same only for cylinders of the same type placed in a column normal to the applied field. In equations (1) and (2), also note that the cosines of the even multiples have been ignored. This is because of anti-symmetric behaviour of the temperature around $\theta = \pi/2$, where θ is measured from the parallel direction.

At the surface of the cylinders, the temperature and the normal component of heat flux are continuous, i.e.

$$T_i = T_{m,i}, \qquad k_i \frac{\partial T_i}{\partial r} = \frac{\partial T_{m,i}}{\partial r} \qquad r = a_i. \quad (3)$$

By applying the above-mentioned boundary conditions, we can obtain

$$A_{2n-1,i} = \frac{B_{2n-1,i}}{\gamma_i a_i^{4n-2}}, \quad (4)$$

$$C_{2n-1,i} = \frac{B_{2n-1,i}}{\chi_i a_i^{4n-2}}, \quad (5)$$

where

$$\gamma_i = \frac{1-k_i}{1+k_i}, \qquad \chi_i = \frac{1-k_i}{2}. \quad (6)$$

For the case of non-conducting cylinders, $k_i = 0$, we have $\gamma_i = 1$ and $\chi_i = 0.5$. Also, applying perfectly conducting cylinders, $k_i = \infty$, implies that $\gamma_i = -1$ and $\chi_i = -\infty$. The unknown coefficients still cannot be determined since the relations given in equations (4) and (5) do not provide a complete set of equations in terms of the unknowns; we require a further series of relations between the coefficients. For this purpose, we employ Rayleigh's strategy which is based on the fact that at any point the temperature may be regarded due to external sources and multiple sources placed at the centre of the cylinders. By examination of temperature function (2) written for a cylinder, we find that terms with radius raised to a positive power cannot be due to sources placed at the centre of that cylinder since they increase when r increases; therefore they stem from the external field and sources originated from the centre of other cylinders. As a result, we can write

$$A_{0,i} + \sum_{n=1}^{\infty} A_{2n-1,i} r^{2n-1} \cos(2n-1)\theta = x + \sum_{j\neq 0} \sum_{n=1}^{\infty} \frac{B_{2n-1,i}}{r_{j,i}^{2n-1}} \cos(2n-1)\theta_{j,i} + \sum_{j} \sum_{n=1}^{\infty} \frac{B_{2n-1,2-\delta_{i2}}}{r_{j,2-\delta_{i2}}^{2n-1}} \cos(2n-1)\theta_{j,2-\delta_{i2}}, \quad (7)$$

where $r_{j,i}$ and $\theta_{j,i}$ are measured from the centre of cylinder j situated in the array of cylinders of type i. As is specified in equation (7), in the sum over the cylinders of type i, all the cylinders, except the cylinder under study ($j = 0$), should be taken into account but in the sum over the cylinders of the other type, all the cylinders are to be considered without any exception. The above expression can be considered as the real part of the following relation:

$$A_{0,i} + \sum_{n=1}^{\infty} A_{2n-1,i}[x - \xi_{0,i} + \mathrm{i}(y - \eta_{0,i})]^{2n-1} = x + \mathrm{i}y + \sum_{j\neq 0} \sum_{n=1}^{\infty} B_{2n-1,i}[x - \xi_{0,i} - \xi_{j,i} + \mathrm{i}(y - \eta_{0,i} - \eta_{j,i})]^{-2n+1} + \sum_{j} \sum_{n=1}^{\infty} B_{2n-1,2-\delta_{i2}} \times [x - \xi_{0,i} - \xi_{j,2-\delta_{i2}} + \mathrm{i}(y - \eta_{0,i} - \eta_{j,2-\delta_{i2}})]^{-2n+1}, \quad (8)$$

where $\xi_{j,i}$ and $\eta_{j,i}$ are the coordinates of cylinder j of type i in the coordinate system (x, y). Now we perform successive differentiation with respect to x on both sides of the above equation and evaluate the results at the centre of the cylinder under study $(\xi_{0,i}, \eta_{0,i})$. After applying equation (4), the process yields the following set of linear algebraic equations in the unknowns $B_{2n-1,i}$ $(i = 1, 2)$:

$$\frac{B_{2n-1,i}}{\gamma_i a_i^{4n-2}} + \sum_{m=1}^{\infty} \binom{2n+2m-3}{2n-1} (S_{2n+2m-2,1} B_{2m-1,i} + S_{2n+2m-2,2} B_{2m-1,2-\delta_{i2}}) = \delta_{n1}, \tag{9}$$

where $S_{2l,i} = \sum_{j\neq 0} (\xi_{j,i} + \mathrm{i}\eta_{j,i})^{-2l}$ are lattice sums over cylinders of type i. Solving equation (9) and using equations (4) and (5) we can obtain all the unknown coefficients and as a result the temperature functions. Since considering $B_{2n-1,i} (i = 1, 2)$ for a sufficiently large n has no significant effect on the values of the temperature functions, in practice, the system of equation (9) is truncated.

3. Determining the effective conductivity of the system

Based on Fourier's law, the effective conductivity of the system can be derived using the following formula [11]:

$$\langle \boldsymbol{F} \rangle = -k_{\mathrm{e}} \langle \nabla T \rangle, \tag{10}$$

where

$$\langle \boldsymbol{F} \rangle = \left(\frac{1}{V_{\mathrm{cell}}}\right) \int_{V_{\mathrm{cell}}} \boldsymbol{F} \, \mathrm{d}V,$$
$$\langle \nabla T \rangle = \left(\frac{1}{V_{\mathrm{cell}}}\right) \int_{V_{\mathrm{cell}}} \nabla T \, \mathrm{d}V,$$

are the average heat flux and temperature gradient over the unit cell, respectively. To proceed, let us decompose the average heat flux as the following:

$$\langle \boldsymbol{F} \rangle = \frac{1}{V_{\mathrm{cell}}} \left[\int_{V_1} \boldsymbol{F} \, \mathrm{d}V + \int_{V_2} \boldsymbol{F} \, \mathrm{d}V + \int_{V_m} \boldsymbol{F} \, \mathrm{d}V \right], \tag{11}$$

whence

$$\langle \boldsymbol{F} \rangle = \frac{1-k_1}{V_{\mathrm{cell}}} \int_{V_1} \nabla T_1 \, \mathrm{d}V + \frac{1-k_2}{V_{\mathrm{cell}}} \int_{V_2} \nabla T_2 \, \mathrm{d}V - \frac{1}{V_{\mathrm{cell}}} \int_{V_{\mathrm{cell}}} \nabla T \, \mathrm{d}V, \tag{12}$$

where V_1, V_2 and V_m are the volumes of cylinders of type one and two and the matrix placed in the unit cell, respectively. After performing the integrals (see appendix A), from equation (12) we may find

$$\langle \boldsymbol{F} \rangle = \frac{2\pi B_{1,1}}{V_{\mathrm{cell}}} \boldsymbol{i} + \frac{2\pi B_{1,2}}{V_{\mathrm{cell}}} \boldsymbol{i} - \langle \nabla T \rangle. \tag{13}$$

Taking into account that $\langle \nabla T \rangle = \boldsymbol{i}$ and $V_{\mathrm{cell}} = b$, the final result for the effective conductivity from equations (13) and (10) can be given as

$$k_{\mathrm{e}} = 1 - \frac{2\pi (B_{1,1} + B_{1,2})}{b} = 1 - 2 f_1 \gamma_1 A_{1,1} + 2 f_2 \gamma_2 A_{1,2}. \tag{14}$$

As can be seen from equation (14), knowing $B_{1,1}$ and $B_{1,2}$ is enough for obtaining the effective conductivity of the system. The term $2\pi(B_{1,1} + B_{1,2})/b$ was produced because of the presence of the inclusions in the matrix. It can be positive (impairing case), negative (enhancing case) or equal to zero. For the case in which N types of cylinders are placed in the unit cell, the effective conductivity of the system can be obtained using

$$k_{\mathrm{e}} = 1 - 2\pi \sum_{i=1}^{N} \frac{B_{1,i}}{V_{\mathrm{cell}}}. \tag{15}$$

4. The effective conductivity for the perpendicular direction

For deriving the effective conductivity in this direction in order to make it similar with the foregone relations, we rotate the system through an angle of $\alpha = \pi/2$. If we follow the above-mentioned procedures for the parallel direction, we get

$$\frac{B'_{2n-1,i}}{\gamma_i a_i^{4n-2}} + \sum_{m=1}^{\infty} \binom{2n+2m-3}{2n-1} (S'_{2n+2m-2,1} B'_{2m-1,i} + S'_{2n+2m-2,2} B'_{2m-1,2-\delta_{i2}}) = \delta_{n1}, \tag{16}$$

where $S'_{2l,i}$ are lattice sums over cylinders of type i in this new position. It can be proved that $S'_{2l,i} = (-1)^l S_{2l,i}$ for $l > 1$ and also $S'_{2,i} = 2\pi/b - S_{2,i}$ (see [6, 12]). The effective conductivity of the system may be obtained using the similar formula as (14), i.e.

$$k'_{\mathrm{e}} = 1 - \frac{2\pi (B'_{1,1} + B'_{1,2})}{b}. \tag{17}$$

Since the composite under study is a two-dimensional structure, through a methodology (see appendix B) we can show that the effective conductivity in the perpendicular direction has been linked to that in the parallel direction using the well-known reciprocal theorem of Keller [13–15], i.e.

$$k_{\mathrm{e}}(k_1, k_2, 1) \times k'_{\mathrm{e}} \left(\frac{1}{k_1}, \frac{1}{k_2}, 1 \right) = 1. \tag{18}$$

5. Explicit solutions

For low-volume fractions or when the conductivity of the cylinders is small, considering a few numbers of $B_{2n-1,1}$ and $B_{2n-1,2}$ in the process of solving (9) may yield reasonable results for $B_{1,1}$ and $B_{1,2}$ and as a result, based on equation (14), for the effective conductivity. It is more useful that, within these boundaries, we manage to derive an explicit relation for the effective conductivity of the system. Based on the method used for truncating (e.g. square or triangular manner) and on the number of the unknowns taken into account, one may obtain different expressions. If we truncate equation (9) in a triangular manner and keep only the coefficient $B_{1,i}$ and $B_{3,i}$ $(i = 1, 2)$, we find

$$\begin{aligned}
&\frac{B_{1,1}}{\gamma_1 a_1^2} + S_{2,1} B_{1,1} + S_{2,2} B_{1,2} + 3 S_{4,1} B_{3,1} + 3 S_{4,2} B_{3,2} = 1, \\
&\frac{B_{3,1}}{\gamma_1 a_1^6} + S_{4,1} B_{1,1} + S_{4,2} B_{1,2} = 0, \\
&\frac{B_{1,2}}{\gamma_2 a_2^2} + S_{2,1} B_{1,2} + S_{2,2} B_{1,1} + 3 S_{4,1} B_{3,2} + 3 S_{4,2} B_{3,1} = 1, \\
&\frac{B_{3,2}}{\gamma_2 a_2^6} + S_{4,1} B_{1,2} + S_{4,2} B_{1,1} = 0.
\end{aligned} \tag{19}$$

Deriving $B_{1,1}$ and $B_{1,2}$ from the above equation and using equation (14), we can obtain the effective conductivity of the system in an explicit relation, i.e.

$$k_e = 1 - \frac{2f_1}{(\lambda_1\lambda_2 - \xi_1\xi_2)/(\lambda_2 - \xi_2)} - \frac{2f_2}{(\lambda_1\lambda_2 - \xi_1\xi_2)/(\lambda_1 - \xi_1)} \tag{20}$$

with

$$\lambda_i = \frac{1}{\gamma_i} + c_1 f_i - c_2\gamma_i f_i^4 - c_3\gamma_{2-\delta_{i2}} f_i f_{2-\delta_{i2}}^3, \tag{21}$$

$$\xi_i = c_4 f_i - c_5(\gamma_i f_i^4 + \gamma_{2-\delta_{i2}} f_i f_{2-\delta_{i2}}^3), \tag{22}$$

where

$$c_1 = S_{2,1}\frac{b}{\pi}, \qquad c_2 = 3\left(\frac{b}{\pi}\right)^4 S_{4,1}^2, \qquad c_3 = 3\left(\frac{b}{\pi}\right)^4 S_{4,2}^2,$$
$$c_4 = S_{2,2}\frac{b}{\pi} \qquad \text{and} \qquad c_5 = 3S_{4,1}S_{4,2}\left(\frac{b}{\pi}\right)^4.$$

One may leave the higher orders and obtain a simpler relation

$$k_e = 1 - \frac{2f_1}{\omega/\upsilon_2} - \frac{2f_2}{\omega/\upsilon_1}, \tag{23}$$

where

$$\omega = \left(\frac{1}{\gamma_1} + c_1 f_1\right)\left(\frac{1}{\gamma_2} + c_1 f_2\right) - c_4^2 f_1 f_2, \tag{24}$$

$$\upsilon_i = \frac{1}{\gamma_i} + (c_1 - c_4) f_i. \tag{25}$$

For the case of uniform cylinders ($f_1 = f_2 = f^*$, $\gamma_1 = \gamma_2 = \gamma^*$) from equation (20), we obtain the following formula:

$$k_e = 1 - \frac{2F}{1/\gamma^* + F - d\gamma^* F^4}, \tag{26}$$

where $F = f_1 + f_2 = 2f^*$ is the total volume fraction and $d_1 = 3(b/\pi)^4(S_{4,1} + S_{4,2})^2/16$. We have listed $c_1, \ldots, c_5$ for the case $b = \sqrt{3}$ in table 1, calculating highly accurate values for lattice sums using integral representation technique [16]. For this case, $d = 0$ as we expected [6]. Note that equation (20) can also be applied for the perpendicular direction if we calculate the coefficients for this direction. In table 1 we have also reported these values.

6. Results and discussion

Before starting the discussion on the results of the three-phase system, it is helpful to verify the validity of the extension to the three-phase system. We perform a series of calculations for two-phase composites with uniform cylinders arranged either in square ($b = 1$) or hexagonal orders ($b = \sqrt{3}$) and then compare the results with those reported by Perrins *et al* [6]. The two-phase cases can be constructed from the three-phase one simply by applying $f_1 = f_2$ and $k_1 = k_2$. For this purpose highly accurate values for lattice sums over cylinders of type one and two were derived and equation (9) ($i = 1, 2$) was solved numerically using LU decomposition method [17]. Taking into account 100 unknowns of $B_{2n-1,1}$ and $B_{2n-1,2}$ gives us a measure of obtaining accurate results for all the volume fractions and conductivities considered (see [6]). Table 2 shows a part of the calculated lattice sums used in the procedure of the solutions. In figure 2 the results are compared for both the square and hexagonal arrays for the most challenging case, i.e. the case of perfectly conducting cylinders. As can be seen for all the values of volume fractions, the results of the two studies are in excellent agreement.

Table 1. The calculated values for $c_1, \ldots, c_5$ used in the analytical formula (20) for determining the effective conductivity in the parallel and perpendicular directions for the case $b = \sqrt{3}$.

	Parallel	Perpendicular
c_1	0.187 018 134	1.812 981 866
c_2	1.310 523 128	1.310 523 128
c_3	1.310 523 128	1.310 523 128
c_4	1.812 981 866	0.187 018 134
c_5	−1.310 523 128	−1.310 523 128

Figure 3 shows a typical result for the effective conductivity of the system for both the parallel and perpendicular directions. The volume fractions are $f_1 = 0.4$ and $f_2 = 0.4$, and the periodicity in the x-direction was supposed to be $b = \sqrt{3}$. For deriving the conductivity of the system in the perpendicular direction we can either solve equation (16) and apply equation (17) or, alternatively, use the Keller theorem for this purpose. Through a careful examination of this figure, it appears that increasing or decreasing the conductivity of both the types of cylinders may enhance or diminish the conductivity of the system, respectively, which is obvious and remains correct for both directions. Furthermore, the system demonstrates higher effective conductivity in the perpendicular direction. This behaviour is a consequence of the rectangular shape of the unit cell which provides a more (less) important role for the cylinders with lower conductivity in the parallel (perpendicular) direction. Interestingly, for the case of mono-sized cylinders with $k_1 = \infty$ and $k_2 = 0$, increasing the volume fraction of the cylinders causes the conductivity of the system to approach zero in the parallel direction and approach infinity in the perpendicular direction (see table 3). When perfectly insulating cylinders touch each other, they form a barrier which prevents heat flow in the parallel direction (note that for the case of spherical inclusions this behaviour does not hold true since heat can pass through the gaps between spheres). This behaviour can also be observed for all systems for which $b > 1$. For the case $b = 1$, however, the system is isotropic and the same results can be expected for both directions. For this case, the same type cylinders are not able to touch each other and a limited value for the effective conductivity of the system can be expected. Surprisingly, we found that the effective conductivity of the system is simply a unity. This result can be confirmed using the Keller theorem as follows: since the system is isotropic and interchange between the material of the cylinders keeps the system unchanged, using equation (18) we can get

$$k_e\left(k, \frac{1}{k}, 1\right) \times k_e'\left(\frac{1}{k}, k, 1\right) = k_e\left(k, \frac{1}{k}, 1\right) \times k_e\left(k, \frac{1}{k}, 1\right) = 1. \tag{27}$$

Table 2. The calculated lattice sums over cylinders of type one ($S_{n,1}$) and two ($S_{n,2}$) for the case $b = \sqrt{3}$ for $n \leqslant 40$. The fourth column shows that the sum of these two lattice sums is zero for cases $n \neq 2$ and $n \neq 6m$ ($m = 1, \ldots, \infty$).

n	$S_{n,1}$	$S_{n,2}$	$S_n = S_{n,1} + S_{n,2}$
2	0.339 213 371 863 0	3.288 385 356 605 4	3.627 598 728 468 4
4	2.174 403 848 897 3	−2.174 403 848 897 3	0.0
6	−2.015 417 144 615 0	−3.847 614 548 810 4	−5.863 031 693 425 3
8	2.026 299 470 614 1	−2.026 299 470 614 1	0.0
10	−1.991 968 543 811 1	1.991 968 543 811 1	0.0
12	2.004 191 455 405 0	4.005 448 516 292 6	6.009 639 971 697 7
14	−1.999 081 877 790 2	1.999 081 877 790 2	0.0
16	2.000 305 162 759 2	−2.000 305 162 759 2	0.0
18	−1.999 921 376 863 0	−3.999 796 979 507 5	−5.999 718 356 370 5
20	2.000 033 868 833 5	−2.000 033 868 833 5	0.0
22	−1.999 988 707 980 1	1.999 988 707 980 1	0.0
24	2.000 004 120 945 6	4.000 007 526 634 1	6.000 011 647 579 8
26	−1.999 998 745 589 4	1.999 998 745 589 4	0.0
28	2.000 000 418 152 9	−2.000 000 418 152 9	0.0
30	−1.999 999 866 203 8	−3.999 999 721 231 7	−5.999 999 587 435 6
32	2.000 000 046 461 1	−2.000 000 046 461 1	0.0
34	−1.999 999 984 513 0	1.999 999 984 513 0	0.0
36	2.000 000 005 249 7	4.000 000 010 324 7	6.000 000 015 574 4
38	−1.999 999 998 279 2	1.999 999 998 279 2	0.0
40	2.000 000 000 573 6	−2.000 000 000 573 6	0.0

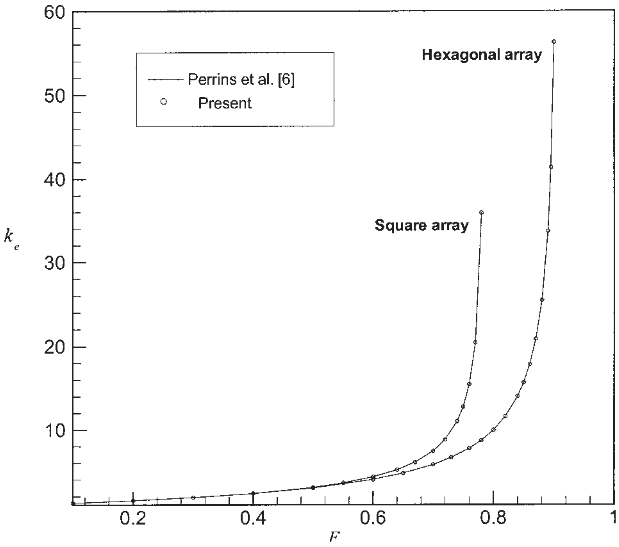

Figure 2. The effective conductivity of two-phase composites with mono-sized perfectly conducting circular cylinders arranged in square and hexagonal orders. F shows the volume fraction of the cylinders. The solid lines are the results of Perrins *et al* [6] and the dotted lines are those obtained by solving the governing equations of the three-phase composites.

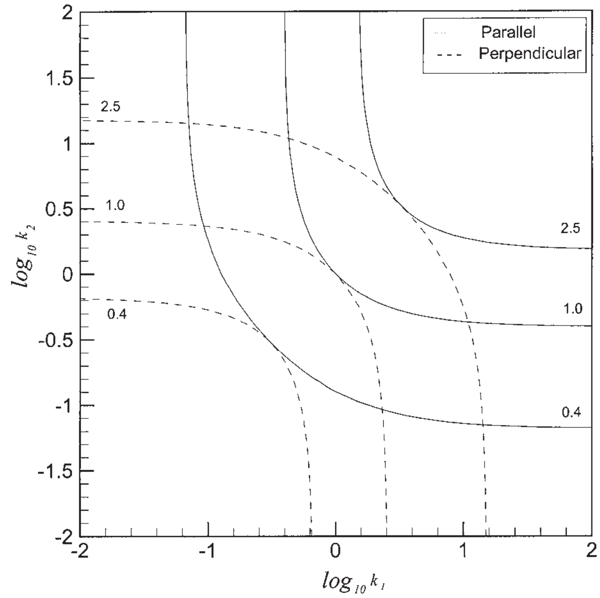

Figure 3. The contours of the effective conductivity in the parallel and perpendicular directions for the case of equal sized cylinders. $f_1 = 0.4$, $f_2 = 0.4$ and $b = \sqrt{3}$.

Considering $k = 0$ proves our case. Sculgasser [18] has shown that in a three-phase system with interchangeable phases (see figure 4), when one of the phases has conductivity equal to k and the two remaining phases are perfectly conducting and non-conducting, the effective conductivity of the system is k. From the above results, it is clear that it is not necessary for the first phase to be interchangeable, and it can simply be a matrix.

For the case $b = 1$ with non-equal sized cylinders, if $f_i \leqslant (\pi/4)(\sqrt{2} - 1)^2$, $f_{2-\delta_{i2}}$ can be increased freely to the touching value limit and the effective conductivity of the system can approach infinity or zero, depending on the conductivity of the touching cylinders.

Figure 5 shows the results of the effective conductivity for a system with a lower total volume fraction, i.e. $f_1 = 0.4$ and $f_2 = 0.2$. b as before is equal to $\sqrt{3}$. Comparing with figure 4 we can see that the case $k_1 = k_2 = 1$ is the only situation in which both systems for the given conductivities present the same effective conductivity. In this situation $\gamma_1 = \gamma_2 = 0$, which leads to $B_{1,1} = B_{1,2} = 0$, and as a result, $k_e = 1$. Figures 3 and 5 also show that having cylinders with conductivities equal to the conductivity of the matrix is not the only condition for the effective conductivity to be equal to the conductivity of the matrix. In fact, this case is a special

Table 3. The effective conductivity in the parallel and perpendicular directions, with respect to the total volume fraction (F). The cylinders are mono-sized, $k_1 = 0$, $k_2 = \infty$ and $b = \sqrt{3}$ ($b > 1$).

F	k_e	k'_e
0.1	0.983 871	1.016 39
0.2	0.937 008	1.067 23
0.3	0.863 599	1.157 94
0.4	0.769 417	1.299 69
0.5	0.660 207	1.514 68
0.6	0.539 863	1.852 32
0.7	0.408 730	2.446 60
0.8	0.261 159	3.829 08
0.9	3E−7	19.253
$\pi/(2\sqrt{3})$	0	—

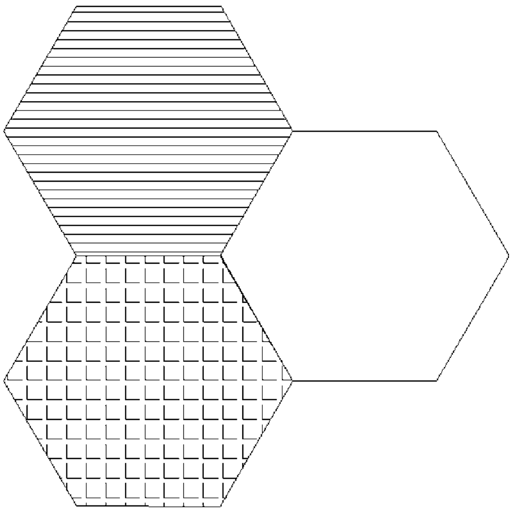

Figure 4. The three-phase structure investigated by Schulgasser *et al* [18]. All the phases are interchangeable.

state of the following general situation:

$$B_{1,1} + B_{1,2} = f_1\gamma_1 A_{1,1} + f_2\gamma_2 A_{1,2} = 0. \qquad (28)$$

The importance of the situation $k_1 = k_2 = 1$ ($B_{1,1} = B_{1,2} = 0$) is that it is independent of the values of f_1, f_2 and b, and for all these situations, we would find that $k_e = 1$, which is physically obvious. This behaviour does not hold for the other values of the conductivities.

7. Conclusion

The effective conductivity of three-phase composites with circular cylindrical inclusions in a periodic arrangement was derived by extending a method put forward by Lord Rayleigh [1]. Considering the recent development in the fast and accurate calculation of lattice sums [16, 19], such an extension can be used efficiently for calculating the conductivity of the system. A study of the behaviour of the composites revealed that they may exhibit unexpected results in particular states. The structure considered in this paper was an ideal one, but the results can be useful for understanding the interplay between microstructures and the effective property of real multi-phase fibre composites and specifically those which can be approximated with the use of periodic structures and circular cylindrical inclusions. Also, the results provide a helpful resource in the process of testing and developing well-known classical numerical methods such as the boundary element method [20] or other proposed schemes in calculating the effective conductivity of multi-phase composite materials.

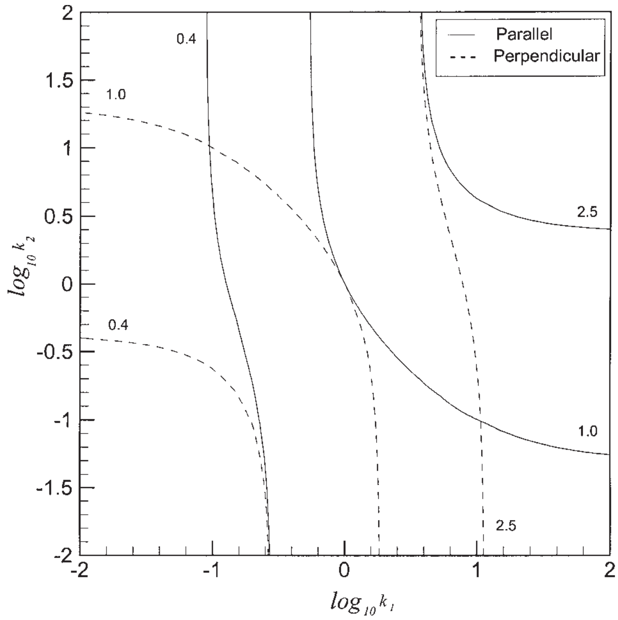

Figure 5. The contours of the effective conductivity in the parallel and perpendicular directions for the case of unequal sized cylinders. $f_1 = 0.4$, $f_2 = 0.2$ and $b = \sqrt{3}$.

Acknowledgments

The authors wish to acknowledge helpful discussion with Professor J Huang regarding calculation of lattice sums.

Appendix A

We would like to calculate the following term:

$$\langle \boldsymbol{S} \rangle_i = \frac{1 - k_i}{V_{\text{cell}}} \int_{V_i} \nabla T_i \, \mathrm{d}V. \qquad \text{(A1)}$$

By using Green's first identity [21], the above relation becomes

$$\langle \boldsymbol{S} \rangle_i = \frac{1 - k_i}{V_{\text{cell}}} \int_{\sigma_i} T_i \boldsymbol{n} \, \mathrm{d}S, \qquad \text{(A2)}$$

where σ_i is the surface of the cylinder of type i in the unit cell and $\boldsymbol{n}$ expresses the unit outward normal vector to the surface. Taking into account the temperature function given in equation (1) and after using the orthogonality properties of trigonometric functions, we can obtain

$$\langle \boldsymbol{S} \rangle_i = \frac{1 - k_i}{V_{\text{cell}}} \int_0^{2\pi} a_i^2 C_{1,i} \cos^2\theta \, \mathrm{d}\theta \, \boldsymbol{i} = \frac{\pi a_i^2 (1 - k_i)}{V_{\text{cell}}} C_{1,i} \boldsymbol{i}$$
$$= \frac{2\pi B_{1,i}}{V_{\text{cell}}} \boldsymbol{i}. \qquad \text{(A3)}$$

Appendix B

Here extending the procedure given by Perrins *et al* [6] we prove Keller's theorem for the system under study as follows: considering that reversing the conductivity of the phases only makes the sign of γ_i $(i = 1, 2)$ negative and applying the mentioned property of lattice sums in section 4, from equation (16), we may write

$$\frac{(-1)^n B'_{2n-1,i}}{\gamma_i a_i^{4n-2}} + \sum_{m=1}^{\infty} \binom{2n+2m-3}{2n-1} (S_{2n+2m-2,1}(-1)^m B'_{2m-1,i} + (-1)^m S_{2n+2m-2,2} B'_{2m-1,2-\delta_{i2}}) = \left[1 - \frac{2\pi(B'_{1,i} + B'_{1,2-\delta_{i2}})}{b}\right]\delta_{n1}. \quad \text{(B1)}$$

Using equation (17) and comparing the above relation with equation (9), we find that

$$\frac{B'_{2n-1,i}}{k'_e(1/k_1, 1/k_2, 1)} = (-1)^n B_{2n-1,i}. \quad \text{(B2)}$$

Writing the above relation for $n = 1$ and using again equation (17) we have

$$k'_e\left(\frac{1}{k_1}, \frac{1}{k_2}, 1\right) = 1 + \frac{2\pi(B_{1,1} + B_{1,2})k'_e(1/k_1, 1/k_2, 1)}{b}. \quad \text{(B3)}$$

Applying equation (14) gives

$$k_e(k_1, k_2, 1) \times k'_e\left(\frac{1}{k_1}, \frac{1}{k_2}, 1\right) = 1. \quad \text{(B4)}$$

References

[1] Rayleigh L 1892 *Phil. Mag.* **34** 481
[2] Batchelor G K and O'Brien R W 1977 *Proc. R. Soc. Ser.* A **355** 313
[3] Kharadly M M Z and Jackson W 1952 *Proc. Electr. Eng.* **100** 199
[4] Yardley J G, McPhedran R C, Nicorovici N A and Botten L C 1999 *Phys. Rev.* E **60** 6068
[5] Hilfer R and Manwart C 2001 *Phys. Rev.* E **64** 021304
[6] Perrins W T, McPhedran R C and McKenzie D R 1979 *Proc. R. Soc. Ser.* A **369** 207
[7] Manteufel R D and Todreas N E 1994 *Int. J. Heat Mass Transfer* **37** 647
[8] Wu F and Whites K W 2001 *Electromagnetics* **21** 97
[9] Batchelor G K 1974 *Ann. Rev. Fluid Mech.* **6** 227
[10] Torquato S 2002 *Random Heterogeneous Materials: Microstructure and Macroscopic Properties* (New York: Springer) p 8
[11] Cheng H and Torquato S 1997 *Proc. R. Soc. Ser.* A **453** 145
[12] Nicorovici N A and McPhedran R C 1996 *Phys. Rev.* E **54** 945
[13] Keller J B 1964 *J. Math. Phys.* **5** 548
[14] Milton G B 1988 *Phys. Rev.* B **38** 11296
[15] Fel L G, Machavariani V S and Bergman D J 2000 *J. Phys. A: Math. Gen.* **33** 6669
[16] Huang J 1999 *J. Math. Phys.* **40** 5240
[17] Press W H, Flannery B P, Tuekolsky S A and Vetterling W T 1986 *Numerical Recipes* (Cambridge: Cambridge University Press) p 42
[18] Schulgasser K 1992 *Int. Comm. Heat Mass Transfer* **19** 639
[19] Movchan A B, Nicorovici N A and McPhedran R C 1997 *Proc. R. Soc. Ser.* A **453** 643
[20] Baltz B A and Ingber M S 1997 *Eng. Anal. Bound. Elem.* **19** 3
[21] Spiegel M R 1968 *Mathematical Handbook of Formulas and Tables* (New York: McGraw-Hill) p 124