Introduction to Elementary Particle Physics

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Elementary Particle Physics

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Bosons and Fermions in Relativistic Quantum Mechanics

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Review of lecture 6 and some remarks:

- Schrödinger equation for a free particle in non-relativistic QM:

$$i\partial_0\psi = -\frac{\boldsymbol{\nabla}^2}{2m}\psi$$

- Klein-Gordon equation for a free (massive) relativistic boson (spin 0 and electrically neutral):

$$\left(\Box + m^2\right)\varphi = 0, \qquad \text{with} \qquad \Box \equiv \partial_0^2 - \boldsymbol{\nabla}^2$$

- Dirac equation for a free (massive) relativistic fermion (spin 1/2 and electrically charged):

$$[i\left(\gamma^0\partial_0 + \gamma^i\partial_i\right) - m]\psi = 0$$

with γ's satisfying the **Clifford-algebra** $(\mu, \nu = 0, 1, 2, 3)$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \text{with} \qquad g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

and
$$\{\gamma^5, \gamma^\mu\} = 0$$

Remark 1:

- Dirac-representation (from lecture 6):

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Weyl-representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

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In general: $i(\gamma^0\partial_0 + \gamma^i\partial_i)\psi(\mathbf{x}, t) = 0$

- Dirac γ matrices:

$$\gamma^0 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}_{4\times4}, \qquad \gamma^i = \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix}_{4\times4}$$

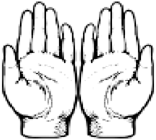
In addition, we define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{4\times4}$$

- γ^5 matrix satisfies the following eigenvalue equation:

$$\gamma^5\,\psi_R = +\psi_R, \qquad \textbf{Right}\text{-handed particles}$$
$$\gamma^5\,\psi_L = -\psi_L \qquad \textbf{Left}\text{-handed particles}$$

- γ^5 is the generator of **chirality** (space-time transformation) !!!



Helicity:

- We start with Dirac equation for **massless** fermions

$$\gamma^0 \partial_0 \psi(\mathbf{x}, t) = -\gamma^i \partial_i \psi(\mathbf{x}, t) = -\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \psi(\mathbf{x}, t)$$

$\times \gamma^5 \gamma^0$ and use

$$(\gamma^0)^2 = 1, \qquad \text{and} \qquad \gamma^5 \gamma^0 \gamma^i \equiv \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}_{4\times 4}$$

to arrive at

$$\gamma^5 \partial_0 \psi(\mathbf{x}, t) = \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \psi(\mathbf{x}, t), \qquad \text{or} \qquad \boxed{\gamma^5 \, p_0 \, \psi = \boldsymbol{\sigma} \cdot \mathbf{p} \, \psi}$$

- For **massless** particles $p_0 = E = \pm |\mathbf{p}|$
- Using $\frac{\mathbf{p}}{p_0} = \pm \frac{\mathbf{p}}{|\mathbf{p}|} \equiv \pm \widehat{\mathbf{p}}$, we obtain $\curvearrowright$

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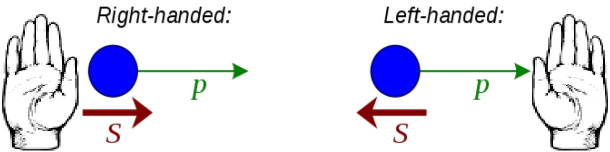
$$\mathcal{H}\psi = \pm\gamma^5\psi, \qquad \text{with} \qquad \mathcal{H} \equiv \boldsymbol{\sigma}\cdot\widehat{\mathbf{p}}$$

Definition: $\mathcal{H} \equiv \boldsymbol{\sigma}\cdot\widehat{\mathbf{p}}$ is the **helicity operator**

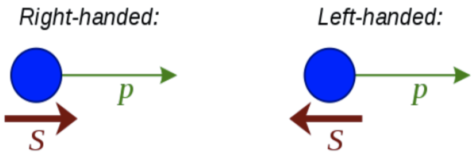
The above relation means

$$\text{Helicity} = \text{sgn(E)} \times \text{Chirality} \rightarrow \begin{cases} \text{Particles} & \text{sgn}(E) > 0 \\ \text{Anti-Particles} & \text{sgn}(E) < 0 \end{cases}$$

For Particles: Helicity=Chirality



- <u>Chirality/Helicity:</u>



$$H = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$$

Helicity= Chirality x Sgn(E)

Chirality		Sgn(E)		Helicity
RH	+	+	Particle	+
LH	-	+		-
RH	+	-	Antiparticle	-
LH	-	-		+

Remark 2:

- Using Weyl-representation for γ-matrices, the Dirac-equation for massless fermions $\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$ reduces to

$$\begin{pmatrix} 0 & E - \boldsymbol{\sigma}\cdot\mathbf{p} \\ E + \boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{cases} \boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi = +E\varphi & \stackrel{E=\pm|\mathbf{p}|}{\longrightarrow} \quad \boldsymbol{\sigma}\cdot\widehat{\mathbf{p}}\,\varphi = \pm\varphi, \\ \boldsymbol{\sigma}\cdot\mathbf{p}\,\chi = -E\chi & \stackrel{E=\pm|\mathbf{p}|}{\longrightarrow} \quad \boldsymbol{\sigma}\cdot\widehat{\mathbf{p}}\,\chi = \mp\chi \end{cases}$$

- On the other hand

$$\gamma^5\psi = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = \begin{pmatrix} -\chi \\ +\varphi \end{pmatrix},$$

$\Longrightarrow$
- χ is a LH **particle** or **antiparticle**
- φ is a RH **particle** or **antiparticle**

Particles and antiparticles have opposite chirality

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Remark 3: New interpretation for $E < 0 \rightarrow$ Antiparticle

- Energy dispersion relation of a relativistic free boson or fermion

$$E^2 = \mathbf{p}^2 + m^2 \Longrightarrow E = \pm\sqrt{\mathbf{p}^2 + m^2}, \qquad \text{Dirac sea} \curvearrowright \gamma \rightarrow e^+ + e^-$$

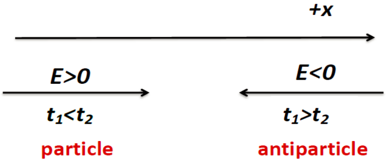
- Feynman-Stueckelberg interpretation:

$$\psi = Ae^{iEt - i\mathbf{p}\cdot\mathbf{x}}$$

  - Wave function ψ represents a **particle** with postive energy $(+|E|)$ and momentum $+\mathbf{p}$ traveling in positive $\mathbf{x}$ direction forwards in time $(t > 0)$.
  - Same wave function ψ represents a **particle** with negative energy $(-|E|)$ and momentum $-\mathbf{p}$ traveling in negative $\mathbf{x}$ direction backwards in time $(t < 0) \longrightarrow$ An **antiparticle**

$$\psi = Ae^{i(-E)(-t) - i(-\mathbf{p})\cdot(-\mathbf{x})} = Ae^{iEt - i\mathbf{p}\cdot\mathbf{x}}$$

Feynman-Stueckelberg interpretation of E$<$0



- **Pair creation** is related to conservation of electric charge (for bosons and fermions): Bosons ($W^{\pm}$), fermions (quarks and leptons)
- The total fermion number should be conserved
- **Question:** Other conservation laws?
  - Lepton flavor number conservation
  - Quark flavor number conservation