Sharif University of Technology - Department of Physics

Quantum Mechanics II - Fall 2021

Problem Set 6

Due Saturday 1400/09/27

Problem 1 (10 pts): Dynamics of a driven two-level system

(a) Let us consider a generic time-dependent Hamiltonian $\hat{H} = \hat{H}_0 + V(t)$. Suppose time-independent states satisfy $\hat{H}_0|n\rangle = E_n|n\rangle$. As you know, in the interaction picture we have $i\hbar\partial_t|\psi(t)\rangle_I = V_I|\psi(t)\rangle_I$, with $V_I(t) = e^{i\hat{H}_0 t/\hbar}V(t)e^{-i\hat{H}_0 t/\hbar}$. If $|\psi(t)\rangle_I = \sum_n c_n(t)|n\rangle$, then show that

$$i\hbar\dot{c}_m(t) = \sum_n V_{mn}(t)e^{i\omega_{mn}t}c_n(t), \tag{1}$$

with $V_{mn}(t) \equiv \langle m|V(t)|n\rangle$ and $\omega_{mn} \equiv (E_m - E_n)/\hbar$ (3 points).

(b) Let us consider a two-state system with

$$\hat{H}_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \qquad V(t) = \begin{pmatrix} 0 & \delta e^{i\omega t} \\ \delta e^{-i\omega t} & 0 \end{pmatrix}. \tag{2}$$

Show that for the two-component vector $\boldsymbol{c}(t) = (c_1(t), c_2(t))$, Eq. (1) translates into

$$i\hbar\partial_t\boldsymbol{c} = \delta\begin{pmatrix} 0 & e^{i(\omega-\omega_{21})t} \\ e^{-i(\omega-\omega_{21})t} & 0 \end{pmatrix}\boldsymbol{c}(t). \qquad \text{(3 points)}$$

(c) Show that for the initial condition $c_1(0) = 1$ and $c_2(0) = 0$, this equation has the solution

$$|c_2(t)|^2 = \frac{\delta^2}{\delta^2 + \hbar^2(\omega-\omega_{21})^2/4}\sin^2\Omega t, \qquad \text{and} \qquad |c_1(t)|^2 = 1 - |c_2(t)|^2,$$

with $\Omega \equiv [(\delta/\hbar)^2 + (\omega-\omega_{21})^2/4]^{1/2}$. Here, Ω is known as the Rabi frequency (3 points).

(d) Show that the maximum probability of occupying state 2 has the value of unity at resonance $\omega = \omega_{21}$ (1 point).

Problem 2 (10 pts): The kicked oscillator

(a) Let us again consider a generic time-dependent Hamiltonian $\hat{H} = \hat{H}_0 + V(t)$. Suppose time-independent states satisfy $\hat{H}_0|n\rangle = E_n|n\rangle$. Consider a system which is prepared in an initial state $|i\rangle$ at time $t = t_0$. As you know, its final state, $|f\rangle$, at a subsequent time, t, is given by $|f\rangle = U_I(t, t_0)|i\rangle$, where the time-evolution operator $U_I(t, t_0)$ satisfies $i\hbar\partial_t U_I(t, t_0) = V_I(t)U_I(t, t_0)$. Show that for $U_I(t_0, t_0) = 1$, we have

$$U_I(t, t_0) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_0}^{t}dt_1\cdots\int_{t_0}^{t_{n-1}}dt_n V_I(t_1)V_I(t_2)\cdots V_I(t_n). \tag{1}$$

(b) Using $1 = \sum_n |n\rangle\langle n|$, we obtain $|f\rangle = \sum_n c_n(t)|n\rangle$ with $c_n(t) = \langle n|U_I(t,t_0)|i\rangle$. Making use of (1) show that $c_n(t) = \sum_{j=0}^{\infty} c_n^{(j)}(t)$, with

$$\begin{aligned} c_n^{(0)} &= \delta_{ni}, \\ c_n^{(1)}(t) &= -\frac{i}{\hbar}\int_{t_0}^{t} dt' e^{i\omega_{ni}t'} V_{ni}(t'), \\ c_n^{(2)}(t) &= -\frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' e^{i\omega_{nm}t' + i\omega_{mi}t''} V_{nm}(t')V_{mi}(t''), \\ \cdots & \quad \cdots \end{aligned} \tag{2}$$

with $V_{mn}(t) \equiv \langle m|V(t)|n\rangle$ and $\omega_{mn} \equiv (E_m - E_n)/\hbar$ (3 points).

(c) Suppose a simple harmonic oscillator is prepared in its ground state $|0\rangle$ at time $t = -\infty$. If it is perturbed by a small time dependent potential $V(t) = -eExe^{-t^2/\tau^2}$. use second order perturbation theory, and determine the probability of finding it in the *second* excited state (4 points).

Problem 3 (10 pts): Alternative derivation of the Golden Rule

(a) Consider $c_n^{(2)}(t)$ from Eq. (2) in problem 2. Suppose that a harmonic potential perturbation $V(t) = e^{\epsilon t} V e^{-i\omega t}$ with the initial time $t_0 \to -\infty$ is gradually switched on. Show that $c_n^{(2)}$ is given by

$$c_n^{(2)}(t) = -\frac{1}{\hbar^2} e^{i(\omega_{ni} - 2\omega)t} \frac{e^{2\epsilon t}}{(\omega_{ni} - 2\omega - 2i\epsilon)} \sum_m \frac{\langle n|V|m\rangle\langle m|V|i\rangle}{\omega_m - \omega_i - \omega - i\epsilon}. \qquad \text{(2 points)}$$

(b) Using $\lim_{\epsilon\to 0} \frac{2\epsilon}{(\omega_{ni}-\omega)^2+\epsilon^2} = 2\pi\delta(\omega_{ni} - \omega)$, show that the transition rate in the limit of $\epsilon \to 0$ is given by

$$\Gamma_{i\to n} = \lim_{\epsilon\to 0} \frac{d|c_n^{(2)}|^2}{dt} = \frac{2\pi}{\hbar^4}\left|\sum_m \frac{\langle n|V|m\rangle\langle m|V|i\rangle}{\omega_m - \omega_i - \omega}\right|^2 \delta(\omega_{ni} - 2\omega). \qquad \text{(2 points)}$$

(c) Describe your interpretation of this result. Do you miss any energy conservation in this transition? (1 points)