HOMEWORK SET 2 - SOLUTIONS

Solution1: Dynamics of a driven two-level system

(a) In the interaction picture we have

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_I = V_I\,|\psi(t)\rangle_I \tag{1}$$

after substituting $|\psi(t)\rangle_I = \sum_n c_n(t)\,|n\rangle$ and $V_I(t) = \exp\left(\frac{iH_0 t}{\hbar}\right)V(t)\exp\left(\frac{-iH_0 t}{\hbar}\right)$, we get

$$LHS = i\hbar\frac{\partial}{\partial t}\Big(\sum_n c_n(t)\,|n\rangle\Big) = i\hbar\sum_n \dot{c}_n(t)\,|n\rangle \tag{2}$$

$$RHS = \exp\left(\frac{iH_0 t}{\hbar}\right)V(t)\exp\left(\frac{-iH_0 t}{\hbar}\right)\sum_n c_n(t)\,|n\rangle = \sum_n \exp\left(\frac{iH_0 t}{\hbar}\right)V(t)\,|n\rangle\exp\left(\frac{-iE_n t}{\hbar}\right)c_n(t) \tag{3}$$

which for the last equality, we used $\exp\left(\frac{-iH_0 t}{\hbar}\right)|n\rangle = \exp\left(\frac{-iE_n t}{\hbar}\right)|n\rangle$. Multiplied both sides by $\langle m|$

$$i\hbar\sum_n \dot{c}_n(t)\,\langle m|n\rangle = \sum_n c_n(t)\exp\left(\frac{-iE_n t}{\hbar}\right)\langle m|\exp\left(\frac{iH_0 t}{\hbar}\right)V(t)\,|n\rangle \tag{4}$$

But we have $\langle m|n\rangle = \delta_{mn}$ and $\langle m|\exp\left(\frac{iH_0 t}{\hbar}\right) = \langle m|\exp\left(\frac{iE_m t}{\hbar}\right)$, thus

$$i\hbar\sum_n \dot{c}_n(t)\delta_{mn} = \sum_n c_n(t)\exp\left(\frac{i(E_m - E_n)t}{\hbar}\right)\langle m|\,V(t)\,|n\rangle \tag{5}$$

with $V_{mn}(t) = \langle m|\,V(t)\,|n\rangle$ and $\omega_{mn} = \frac{E_m - E_n}{\hbar}$ and $\sum_n \dot{c}_n(t)\delta_{mn} = \dot{c}_m(t)$,

$$i\hbar\dot{c}_m(t) = \sum_n V_{mn}(t)\exp(i\omega_{mn}t)c_n(t) \tag{6}$$

(b) Consider the first component $c_1(t)$ and use (6)

$$\begin{aligned} i\hbar\dot{c}_1(t) &= V_{11}(t)\exp(i\omega_{11}t)c_1(t) + V_{12}(t)\exp(i\omega_{12}t)c_2(t) \\ &= 0 + \delta\exp(i(\omega - \omega_{21})t)c_2(t) \\ &= \delta\exp(i(\omega - \omega_{21})t)c_2(t) \end{aligned} \tag{7}$$

Which we have used in the second equality $V_{11}(t) = 0, V_{12}(t) = \delta\exp(i\omega t), \omega_{12} = -\omega_{21}$. Similarly for $c_2(t)$

$$\begin{aligned} i\hbar\dot{c}_2(t) &= V_{21}(t)\exp(i\omega_{21}t)c_1(t) + V_{22}(t)\exp(i\omega_{22}t)c_2(t) \\ &= \delta\exp(-i(\omega - \omega_{21})t)c_1(t) + 0 \\ &= \delta\exp(-i(\omega - \omega_{21})t)c_1(t) \end{aligned} \tag{8}$$

Which we have also used $V_{22}(t) = 0, V_{21}(t) = \delta\exp(-i\omega t)$. From (7) and (8), we get

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \delta\begin{pmatrix} 0 & \exp(i(\omega - \omega_{21})t) \\ \exp(-i(\omega - \omega_{21})t) & 0 \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \tag{9}$$

(c) From equation (9), we have

$$\begin{gathered} i\hbar\tfrac{d}{dt}c_1(t) = \delta\exp(i(\omega - \omega_{21})t)c_2(t) \\ \tfrac{i\hbar}{\delta}\exp(-i(\omega - \omega_{21})t)\tfrac{d}{dt}c_1(t) = c_2(t) \\ \tfrac{i\hbar}{\delta}(-i(\omega - \omega_{21})\exp(-i(\omega - \omega_{21})t)\tfrac{d}{dt}c_1(t) + \exp(-i(\omega - \omega_{21})t)\tfrac{d^2}{dt^2}c_1(t) = \tfrac{d}{dt}c_2(t) \end{gathered} \tag{10}$$

But we know $\frac{d}{dt}c_2(t) = \frac{\delta}{i\hbar}\exp(-i(\omega - \omega_{21})t)c_1(t)$, therefore

$$\frac{d^2}{dt^2}c_1(t) - i(\omega - \omega_{21})\frac{d}{dt}c_1(t) + \frac{\delta^2}{\hbar^2}c_1(t) = 0 \tag{11}$$

Which we have divided both sides by $\exp(-i(\omega-\omega_{21})t)$. The solution of this equation is

$$c_1(t) = A_1\exp(s_1 t) + A_2\exp(s_2 t) \tag{12}$$

Where $s_{1,2} = \frac{i(\omega-\omega_{21})}{2} \pm i\sqrt{(\frac{\delta}{\hbar})^2 + \frac{(\omega-\omega_{21})^2}{4}} = \frac{i(\omega-\omega_{21})}{2} \pm i\Omega$. From initial conditions, we get

$$c_1(0) = 1 \rightarrow A_1 + A_2 = 1 \tag{13}$$

$$c_2(0) = 0 \rightarrow \frac{d}{dt}c_1(t)|_{t=0} = 0 \rightarrow \frac{i(\omega-\omega_{21})}{2}(A_1+A_2) + i\Omega(A_1-A_2) = 0 \tag{14}$$

As a result, we obtain

$$\begin{array}{c} A_1 = \frac{1}{2} - \frac{\omega-\omega_{21}}{4\Omega} \\ A_2 = \frac{1}{2} + \frac{\omega-\omega_{21}}{4\Omega} \end{array} \tag{15}$$

Now, we can compute $|c_1(t)|^2$

$$\begin{aligned} |c_1(t)|^2 &= \frac{1}{2} + \frac{(\omega-\omega_{21})^2}{8\Omega^2} + (\frac{1}{2} - \frac{(\omega-\omega_{21})^2}{8\Omega^2})\cos(2\Omega t) \\ &= 1 - (1 - \frac{(\omega-\omega_{21})^2}{4\Omega^2})\sin^2(\Omega t) \\ &= 1 - \frac{\delta^2}{\delta^2 + \hbar^2\frac{(\omega-\omega_{21})^2}{4}}\sin^2(\Omega t) \end{aligned} \tag{16}$$

But we have ${}_I\langle\psi(t)|\psi(t)\rangle_I = 1 = \sum_{n,m} c_n(t)c_m^*(t)\langle m|n\rangle = \sum_n |c_n(t)|^2$. So $|c_2(t)|^2 = 1 - |c_1(t)|^2$ and finally we have

$$|c_2(t)|^2 = \frac{\delta^2}{\delta^2 + \hbar^2\frac{(\omega-\omega_{21})^2}{4}}\sin^2(\Omega t) \tag{17}$$

(d) We have the maximum probability when $\sin^2(\Omega t_{max}) = 1$, thus at resonance

$$P_{2,max} = \frac{\delta^2}{\delta^2 + \hbar^2\frac{(\omega-\omega_{21})^2}{4}}|_{\omega=\omega_{21}} = 1 \tag{18}$$

Solution2: The kicked oscillator

(a) We know $U_I(t,t_0)$ satisfies

$$\begin{array}{c} i\hbar\partial_t U_I(t,t_0) = V_I(t)U_I(t,t_0) \\ \rightarrow U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^t dt' V_I(t')U_I(t',t_0) \end{array} \tag{19}$$

Which we have used $U_I(t_0,t_0) = 1$ in the above integral form.

$$\begin{aligned} U_I(t,t_0) &= 1 - \frac{i}{\hbar}\int_{t_0}^t dt_1 V_I(t_1)(1 - \frac{i}{\hbar}\int_{t_0}^{t_1} dt_2 V_I(t_2)U_I(t_2,t_0)) \\ &= 1 + (-\frac{i}{\hbar})\int_{t_0}^t dt_1 V_I(t_1) + (-\frac{i}{\hbar})^2\int_{t_0}^t dt_1\int_{t_0}^{t_1} dt_2 V_I(t_1)V_I(t_2)U_I(t_2,t_0) \end{aligned} \tag{20}$$

Substitute $U_I(t_i,t_0)$ in each step and then we have

$$U_I(t,t_0) = \sum_{n=0}^{\infty}(-\frac{i}{\hbar})^n\int_{t_0}^t dt_1 ... \int_{t_0}^{t_{n-1}} dt_n V_I(t_1)V_I(t_2)...V_I(t_n) \tag{21}$$

(b)

$$c_n(t) = \langle n|U_I(t,t_0)|i\rangle = \sum_{j=0}^{\infty}(-\frac{i}{\hbar})^j\int_{t_0}^t dt_1 ... \int_{t_0}^{t_{j-1}} dt_j \langle n|V_I(t_1)V_I(t_2)...V_I(t_j)|i\rangle \tag{22}$$

As a consequent

$$c_n^{(j)}(t) = (-\frac{i}{\hbar})^j \int_{t_0}^{t} dt_1 ... \int_{t_0}^{t_{j-1}} dt_j \langle n| V_I(t_1)V_I(t_2)...V_I(t_j) |i\rangle \tag{23}$$

We can read the coefficients

$$c_n^{(0)}(t) = \langle n|i\rangle = \delta_{ni} \tag{24}$$

$$\begin{aligned} c_n^{(1)}(t) &= \frac{-i}{\hbar} \int_{t_0}^{t} dt_1 \langle n| V_I(t_1) |i\rangle = \frac{-i}{\hbar} \int_{t_0}^{t} dt_1 \langle n| V(t_1) |i\rangle \exp\left(\frac{i(E_n - E_i)t_1}{\hbar}\right) \\ c_n^{(1)}(t) &= \frac{-i}{\hbar} \int_{t_0}^{t} dt^{'} V_{ni}(t^{'}) \exp\left(i\omega_{ni}t^{'}\right) \end{aligned} \tag{25}$$

$$\begin{aligned} c_n^{(2)}(t) &= (\frac{-i}{\hbar})^2 \int_{t_0}^{t} dt^{'} \int_{t_0}^{t^{'}} dt^{''} \langle n| V_I(t^{'})V_I(t^{''}) |i\rangle \\ &= \frac{-1}{\hbar^2} \int_{t_0}^{t} dt^{'} \int_{t_0}^{t^{'}} dt^{''} \sum_m \exp\left(\frac{i(E_n - E_m)t^{'}}{\hbar}\right) \exp\left(\frac{i(E_m - E_i)t^{''}}{\hbar}\right) V_{nm}(t^{'})V_{mi}(t^{''}) \\ &= \frac{-1}{\hbar^2} \int_{t_0}^{t} dt^{'} \int_{t_0}^{t^{'}} dt^{''} \sum_m \exp\left(i\omega_{nm}t^{'} + i\omega_{mi}t^{''}\right) V_{nm}(t^{'})V_{mi}(t^{''}) \end{aligned} \tag{26}$$

(c) The zeroth and first order is zero because

$$c_2^{(0)} = \delta_{20} = 0 \tag{27}$$

$$\begin{aligned} c_2^{(1)} &= \frac{-i}{\hbar} \int_{t_0}^{t} dt^{'} \exp\left(i\omega_{20}t^{'}\right) \langle 2| (-eEx \exp\left(\frac{-t^{'2}}{\tau^2}\right)) |0\rangle \\ &= \frac{i}{\hbar} eE \int_{t_0}^{t} dt^{'} \exp\left(i\omega_{20}t^{'}\right) \exp\left(\frac{-t^{'2}}{\tau^2}\right) \langle 2| x |0\rangle = 0 \end{aligned} \tag{28}$$

Which we have used $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^{\dagger})$ in the last equality. Thus, we have to use the second order perturbation theory and we have

$$\begin{aligned} c_2^{(2)}(t) &= \frac{-1}{\hbar^2} \int_{t_0}^{t} dt^{'} \int_{t_0}^{t^{'}} dt^{''} \sum_m \exp\left(i\omega_{2m}t^{'}\right) \exp\left(i\omega_{m0}t^{''}\right) e^2E^2 \exp\left(\frac{-t^{'2}}{\tau^2}\right) \exp\left(\frac{-t^{''2}}{\tau^2}\right) \langle 2| x |m\rangle \langle m| x |0\rangle \\ &= \frac{-1}{\hbar^2} e^2E^2 \int_{t_0}^{t} dt^{'} \int_{t_0}^{t^{'}} dt^{''} \exp\left(i\omega_{21}t^{'}\right) \exp\left(i\omega_{10}t^{''}\right) \exp\left(\frac{-t^{'2}}{\tau^2}\right) \exp\left(\frac{-t^{''2}}{\tau^2}\right) \frac{\hbar}{2m\omega} \langle 2| a^{\dagger} |1\rangle \langle 1| a^{\dagger} |0\rangle \\ &= \frac{-1}{\hbar^2} e^2E^2 \frac{\hbar}{\sqrt{2}m\omega} \int_{t_0}^{t} dt^{'} \int_{t_0}^{t^{'}} dt^{''} \exp\left(i\omega t^{'}\right) \exp\left(i\omega t^{''}\right) \exp\left(\frac{-t^{'2}}{\tau^2}\right) \exp\left(\frac{-t^{''2}}{\tau^2}\right) \end{aligned} \tag{29}$$

But we know $\int_{t_0}^{t} dt^{'} \int_{t_0}^{t^{'}} dt^{''} \rightarrow \frac{1}{2} \int_{t_0}^{t} dt^{'} \int_{t_0}^{t} dt^{''}$ and finally

$$c_2^{(2)}(t) = \frac{-e^2E^2}{2\sqrt{2}m\omega\hbar} (\int_{t_0}^{t} dt^{'} \exp\left(i\omega t^{'} + \frac{-t^{'2}}{\tau^2}\right))^2 \tag{30}$$

Consider the limit of infinity and this is just the standard Gaussian integral. We obtain

$$c_2^{(2)}(\infty) = \frac{-e^2E^2}{2\sqrt{2}m\omega\hbar} \pi\tau^2 \exp\left(\frac{-\omega^2\tau^2}{2}\right) \tag{31}$$

and also the probability is

$$P = |c_2^{(2)}(\infty)|^2 = \frac{\pi^2 e^4 E^4 \tau^4}{8m^2\omega^2\hbar^2}\exp\left(-\omega^2\tau^2\right) \tag{32}$$

Solution3: Alternative derivation of the golden rule

(a) From the previous problem we have

$$\begin{aligned}
c_n^{(2)} &= \frac{-1}{\hbar^2}\int_{-\infty}^{t} dt^{'} \int_{-\infty}^{t^{'}} dt^{''} \sum_m \exp\left(i\omega_{nm}t^{'}\right)\exp\left(i\omega_{mi}t^{''}\right)\langle n|\,V\,|m\rangle\,\langle m|\,V\,|i\rangle \exp\left(\epsilon t^{'} - i\omega t^{'}\right)\exp\left(\epsilon t^{''} - i\omega t^{''}\right) \\
&= \frac{-1}{\hbar^2}\sum_m \int_{-\infty}^{t} e^{i(\omega_{nm}-\omega)t'+\epsilon t'} dt' \int_{-\infty}^{t'} dt^{''} e^{i(\omega_{mi}-\omega)t^{''}+\epsilon t^{''}}\langle n|V|m\rangle\langle m|V|i\rangle \\
&= \frac{-1}{\hbar^2}\sum_m \int_{-\infty}^{t} e^{i(\omega_{nm}-\omega)t'+\epsilon t'} dt' \frac{1}{i\left(\omega_{mi}-\omega\right)+\epsilon} e^{i(\omega_{mi}-\omega)t'+\epsilon t'}\langle n|V|m\rangle\langle m|V|i\rangle \\
&= \frac{-1}{\hbar^2}\sum_m \frac{1}{i\left(\omega_{mi}-\omega\right)+\epsilon}\,\frac{1}{i\left(\omega_{nm}+\omega_{mi}-2\omega\right)+2\epsilon} e^{i(\omega_{nm}+\omega_{mi}-2\omega)t+2\epsilon t}\langle n|V|m\rangle\,\langle m|V|i\rangle \\
&= \frac{-1}{\hbar^2} e^{i(\omega_{ni}-2\omega)t}\,\frac{i^2 e^{2\epsilon t}}{(2i\epsilon - \omega_{ni}+2\omega)}\sum_m \frac{\langle n|V|m\rangle\langle m|V|i\rangle}{i\epsilon - \omega_m + \omega_i - \omega} \\
&= \frac{1}{\hbar^2} e^{i(\omega_{ni}-2\omega)t}\,\frac{e^{2\epsilon t}}{(\omega_{ni}-2\omega-2i\epsilon)}\sum_m \frac{\langle n|V|m\rangle\langle m|V|i\rangle}{\omega_m - \omega_i - \omega - i\epsilon}
\end{aligned} \tag{33}$$

(b) using (33)

$$\left|C_n^{(2)}(t)\right|^2 = \frac{1}{\hbar^4}\frac{e^{4\epsilon t}}{\left(\left(\omega_{ni}-2\omega\right)^2 + (2\epsilon)^2\right)}\left|\sum_m \frac{\langle n|V|m\rangle\langle m|V|i\rangle}{\omega_m - \omega_i - \omega - i\epsilon}\right|^2$$

Now in the limit $\epsilon \to 0$ and $\lim_{\epsilon\to 0}\frac{4\epsilon}{(\omega_{ni}-2\omega)^2+(2\epsilon)^2} = 2\pi\delta\left(\omega_{ni}-2\omega\right)$, we have

$$\begin{aligned}
&\Gamma_{i\to n} = \lim_{\epsilon\to 0}\frac{d\left|c_n^{(2)}\right|^2}{dt} \\
&\frac{d\left|c_n^{(2)}\right|^2}{dt} = \frac{1}{\hbar^4}\frac{4\epsilon e^{4\epsilon t}}{\left(\omega_{ni}-2\omega\right)^2+(2\epsilon)^2}\left|\sum_m \frac{\langle n|V|m\rangle\langle m|V|i\rangle}{\omega_m-\omega_i-\omega-i\epsilon}\right|^2 \\
&\Rightarrow \lim_{\epsilon\to 0}\frac{d\left|c_n^{(2)}\right|^2}{dt} = \frac{1}{\hbar^4}\lim_{\epsilon\to 0}\frac{4\epsilon\delta\left(\omega_{ni}-2\omega\right)}{\left(\omega_{ni}-2\omega\right)^2+(2\epsilon)^2}\left|\sum_m \frac{\langle n|V|m\rangle\langle m|V|i\rangle}{\omega_m-\omega_i-\omega}\right|^2 \\
&\Longrightarrow \Gamma_{i\to n} = \frac{2\pi}{\hbar^4}\left|\sum_m \frac{\langle n|V|m\rangle\langle m|V|i\rangle}{\omega_m-\omega_i-\omega}\right|^2\delta\left(\omega_{ni}-2\omega\right)
\end{aligned} \tag{34}$$

(c) The transition probability $\Gamma_{i\to n}$ is proportional to the product of two separate transition ($i \to m$ and $m \to n$) probabilities or matrix elements. Therefore, the probability is far less than that for a first order transition since it involves the product of two probabilities.

To determine the overall transition rate we must sum over the complete set of all intermediate states m before squaring. The intermediate states need not be real. The system can make a transition to an intermediate virtual state and then to a final real state. Actually, it is possible to rewrite the equation (34) as

$$\Gamma_{i\to n} = \frac{2\pi}{\hbar}\left|\sum_m \frac{\langle n|V|m\rangle\,\langle m|V|i\rangle}{E_m - E_i - \hbar\omega}\right|^2\delta\left(E_{ni}-2\hbar\omega\right)$$

From $\delta\left(E_{ni}-2\hbar\omega\right)$ we have $E_n = E_i + 2\hbar\omega$. As we've already said, we have (i $\to m$ and m $\to n$) , thus after absorbing two photons from external source $V(t)$, the transition (i $\to n$) will occur.