General-relativistic viscous fluids

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Webinar on quark matter and relativistic hydrodynamics

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Relativistic ideal fluids

A (relativistic) ideal fluid is described by the (relativistic) Euler equations

$$\nabla_\alpha \mathcal{T}^\alpha_\beta = 0,$$
$$\nabla_\alpha J^\alpha = 0,$$

where $\mathcal{T}$ is the energy-momentum tensor of an ideal fluid given by

$$\mathcal{T}_{\alpha\beta} = (p + \varrho) u_\alpha u_\beta + p g_{\alpha\beta},$$

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Above, ϱ is the fluid's (energy) density, n is the baryon density, $p = p(\varrho, n)$ is the fluid's pressure, and u is the fluid's (four-)velocity, which satisfies

$$g_{\alpha\beta} u^\alpha u^\beta = -1.$$

g is the spacetime metric and ∇ the corresponding covariant derivative.

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There are, however, important situations where a theory or relativistic viscous fluids is needed.

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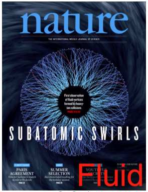

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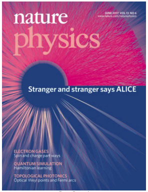

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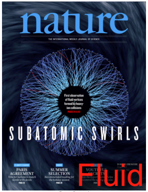

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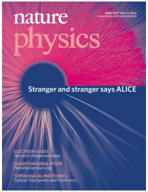

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Theory, experiments, numerical simulation, phenomenology: the QGP is a relativistic liquid with viscosity.

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Conclusion: Knudsen number $K_n \sim \ell/L$ may not be small in some cases $\Rightarrow$ viscous contributions likely to affect the gravitational wave signal.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

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First-order theory: $\pi = \pi(\varrho, u, \partial\varrho, \partial u, \dots)$ etc.

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The Eckart and Landau-Lifshitz theories

Starting from:

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Eckart ('40) and Landau-Lifshitz ('50) (first-order): $\mathcal{R} = 0$,

$$\pi_{\alpha\beta} := -2\eta\Pi^\mu_\alpha\Pi^\nu_\beta(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\nabla_\lambda u^\lambda g_{\mu\nu}),\ \mathcal{P} := -\zeta\nabla_\mu u^\mu,\ (\mathcal{Q}_\alpha = 0),$$

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In essence:

1. Covariant generalization of Navier-Stokes.
2. Entropy production ≥ 0.

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Instability/acausality results apply to large classes of first-order theories. Difficult to construct causal and stable theories of relativistic fluids with viscosity: great deal of work trying to address the issue.

The Israel-Stewart theory

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

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$$\begin{aligned} \tau_{\mathcal{P}} u^\mu \nabla_\mu \mathcal{P} + \mathcal{P} + \zeta \nabla_\mu u^\mu &= \mathcal{J}^{\mathcal{P}}, \\ \tau_\pi u^\mu \Pi^\nu_\alpha \nabla_\mu \mathcal{Q}_\nu + \mathcal{Q}_\alpha &= \mathcal{J}^{\mathcal{Q}}_\alpha, \\ \tau_\pi u^\lambda \widehat{\Pi}^{\mu\nu}_{\alpha\beta} \nabla_\lambda \pi_{\mu\nu} + \pi_{\alpha\beta} - 2\eta\sigma_{\alpha\beta} &= \mathcal{J}^\pi_{\alpha\beta}, \end{aligned}$$

where $\widehat{\Pi}$ is the $u^\perp$ 2-tensor projection onto its symmetric and trace-free part; σ is the $u^\perp$ trace-free part of ∇u, $\tau' s = \tau(\varrho)$ are relaxation times.

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where $\widehat{\Pi}$ is the $u^\perp$ 2-tensor projection onto its symmetric and trace-free part; σ is the $u^\perp$ trace-free part of ∇u, $\tau' s = \tau(\varrho)$ are relaxation times.

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The Israel-Stewart theory

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

Israel-Stewart: second-order theory ('70s). Modern versions: Baier-Romatschke-Son-Starinets-Stephanov ('08); Denicol-Niemi-Molnar-Rischke ('12). EoM: $\mathcal{R} = 0$, $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ and

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System is highly complex; large system with non-diagonal principal part.

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Proof by contradiction: it does not reveal the nature of the singularity; first breakdown result for Israel-Stewart.

The BDNK theory

The BDNK theory is a first-order theory defined by (D-Bemfica-Noronha, ’18, ’19, ’20; Kovtun, ’19; Hoult-Kovtun, ’20):

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Lots of terms: need them to fix the causality and instability problems of Eckart and Landau-Lifshitz. One should let the fundamental principle of causality constrain which terms are allowed in the theory rather than decide the possible terms and then try to establish causality

Theorem: Causality, stability, and LWP of the BDNK theory (D-Bemfica-Rodriguez-Shao, '19; D-Bemfica-Graber, '20; D-Bemfica-Noronha, '20)

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Theorem valid with baryon current and $p = p(\varrho, n)$.

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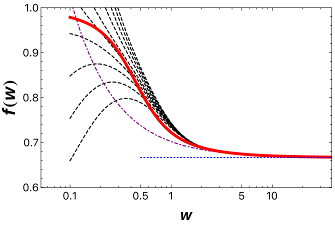

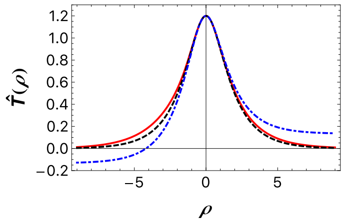

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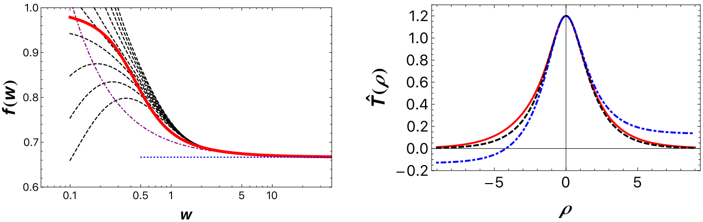

The BDNK theory has all the good features of the Israel-Stewart theory plus a good local well-posedness theory in Sobolev spaces, which is lacking for Israel-Stewart (applications to neutron star mergers).

Entropy production

First law: $T\mathbb{S}^\mu = pu^\mu - u_\nu \mathcal{T}^{\nu\mu} - \mu J^\mu$; $\frac{dp}{\varrho+p} = \frac{dT}{T} + \frac{nT}{\varrho+p} d\left(\frac{\mu}{T}\right)$.

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In a first order approach, $\nabla_\mu \mathbb{S}^\mu$ can only be correctly determined up to $O(\partial^2)$. (Kovtun, '19.)

Entropy production

First law: $T\mathbb{S}^\mu = pu^\mu - u_\nu \mathcal{T}^{\nu\mu} - \mu J^\mu$; $\frac{dp}{\varrho+p} = \frac{dT}{T} + \frac{nT}{\varrho+p} d\left(\frac{\mu}{T}\right)$.

Power counting: $\phi = O(1)$, $\nabla\phi = O(\partial)$, $\nabla^2\phi, (\nabla\phi)^2 = O(\partial^2)$, etc.

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On-shell:

$$\nabla_\mu \mathbb{S}^\mu = 2\eta \frac{\sigma_{\mu\nu}\sigma^{\mu\nu}}{T} + \zeta \frac{(\nabla_\mu u^\mu)^2}{T} + T\left[\Pi^\lambda_\nu \nabla_\lambda \left(\frac{\mu}{T}\right)\right]\left[\Pi^{\nu\alpha}\nabla_\alpha\left(\frac{\mu}{T}\right)\right] + O(\partial^3)$$
$$\gtrsim 0.$$

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– Thank you for your attention –