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CRC–TR 211

Kinetic theory and hydrodynamics for spin-1/2 particles with nonlocal collisions

Nora Weickgenannt

NW, X.-l. Sheng, E. Speranza, Q. Wang, and D.H. Rischke,
PRD 100 (2019) 5, 056018,
NW, E. Speranza, X.-l. Sheng, Q. Wang, and D.H. Rischke,
arXiv:2005.01506 (2020),
E. Speranza and NW, arXiv:2007.00138 (2020)

Seminar at Sharif University of Technology Tehran-Iran

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Overview

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- From quantum field theory to kinetic theory: Wigner functions
- Free-streaming limit:
  Collisionless Boltzmann equation with electromagnetic fields
- Massless limit
- Global equilibrium with standard collision term
- Polarization from vorticity?
- Nonlocal collision term
- Equilibrium with nonlocal collision term
- Spin hydrodynamics and pseudo-gauges
- Nonrelativistic limit

Why magneto-hydrodynamics (MHD)?

- Early stage of non-central heavy-ion collisions: large orbital angular momenta and strong electromagnetic fields.

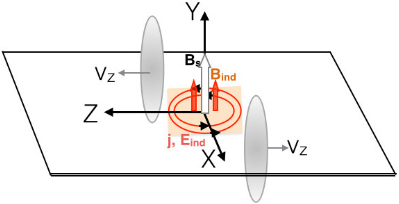

Figure from V. Roy, S. Pu, L. Rezzolla, and D. H. Rischke, PRC96 (2017) 054909

- Strong electromagnetic fields in early universe and compact stars.
- For massive spin-0 particles, second-order dissipative MHD has already been studied.

  G.S. Denicol, X.-G. Huang, E. Molnar, G.M. Monteiro, H. Niemi, J. Noronha, D.H. Rischke, and Q. Wang, PRD 98 (2018) 076009

  G.S. Denicol, E. Molnar, H. Niemi, and D.H. Rischke, PRD 99 (2019) 056017
- But all elementary matter particles are fermions...

Spin effects in heavy-ion collisions

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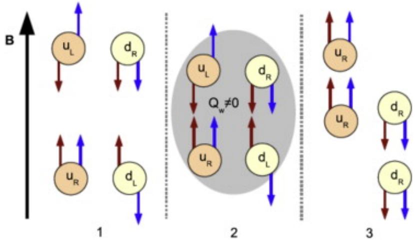
- Chiral vortical effect (CVE): charge currents induced by vorticity.
- Chiral magnetic effect (CME): charge currents induced by magnetic fields.



Dmitri E. Kharzeev, Larry D. McLerran, and Harmen J. Warringa, NPA 803 (2008)

- Has been studied in massless case.

  J-Y. Chen, D. T. Son, and M. Stephanov, PRL 115 (2015), 021601;

  Y. Hidaka, S. Pu, and D-L. Yang, PRD95 (2017) 091901;

  A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98 (2018) 036010
- Similar effects for massive particles?

Towards MHD with spin

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- What we want: kinetic theory and fluid dynamics for massive spin-1/2 particles in inhomogeneous electromagnetic fields.

  J.-H. Gao, and Z.-T. Liang, arXiv:1902.06510 (2019)

  K. Hattori, Y. Hidaka, and D.-L. Yang, arXiv:1903.01653 (2019)

  Z. Wang, X. Guo, S. Shi, and P. Zhuang, arXiv:1903.03461 (2019)
- Starting point: quantum field theory, Dirac equation.
- Strategy: use Wigner functions to derive kinetic theory.
- Goal: determine fluid-dynamical equations of motion from resulting Boltzmann equation.

Wigner functions

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- Quantum analogue of classical distribution function.
- Contains information about quantum state of system.
- Off-equilibrium: two-point function depends not only on relative coordinate y, but also on central coordinate x.
- Wigner transformation of two-point function:
  H.-Th. Elze, M. Gyulassy, and D. Vasak, Ann. Phys. 173 (1987) 462

$$W(x,p) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \langle : \bar{\Psi}(x+\frac{y}{2})\Psi(x-\frac{y}{2}) : \rangle ,$$

Wigner functions

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$$W(x,p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p\cdot y} \left\langle : \bar{\Psi}(x+\frac{y}{2}) U(x+\frac{y}{2}, x) U(x, x-\frac{y}{2}) \Psi(x-\frac{y}{2}) : \right\rangle ,$$

with gauge link

$$U(b,a) \equiv P \exp\left(-\frac{i}{\hbar}\int_a^b dz^\mu \, A_\mu(z)\right)$$

to ensure gauge invariance.

Kinetic equation for Wigner function

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- Dirac equation $\Longrightarrow$ Equation of motion for Wigner function
  S. R. De Groot, W. A. Van Leeuwen, and C. G. Van Weert, Relativistic Kinetic Theory. Principles and Applications (North-Holland, 1980)
  D. Vasak, M. Gyulassy, and H. T. Elze, AP 173, 462 (1987)

$$(\gamma \cdot K - m)W = \hbar \mathcal{C}$$

with

$$\begin{aligned} K^\mu &\equiv \Pi^\mu + \frac{1}{2} i\hbar\nabla^\mu, \\ \nabla^\mu &\equiv \partial_x^\mu - j_0(\Delta)F^{\mu\nu}\partial_{p\nu}, \\ \Pi^\mu &\equiv p^\mu - \hbar\frac{1}{2}j_1(\Delta)F^{\mu\nu}\partial_{p\nu}, \end{aligned}$$

$\Delta = \frac{1}{2}\hbar\partial_p \cdot \partial_x$ with ∂_x only acting on $F^{\mu\nu}$ and $j_0(r) = \sin(r)/r$, $j_1(r) = [\sin(r) - r\cos(r)]/r^2$ spherical Bessel functions.

- Exact quantum kinetic equation for Wigner function for massive spin 1/2-particles and inhomogeneous fields!
- Collision term $\mathcal{C}$

Calculating the Wigner function

- In general: result of calculation of Wigner function directly from definition is not on-shell.
- Momentum variable of directly calculated Wigner function is physical (kinetic) momentum
  for vanishing gradients, i.e., in global equilibrium without rotation,
  or in the classical limit ($\hbar = 0$).
- But we want:
  inhomogeneous phase-space distribution,
  quantum effects.
- Idea: Find general solutions of transport equation for Wigner function by expanding in powers of $\hbar$.
- For zeroth order use results of direct calculation.

Free-streaming limit

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- First: study effects of electromagnetic mean fields, neglect collisions.
- Free streaming $\mathcal{C} = 0$
  NW, X.-L. Sheng, E. Speranza, Q. Wang, and D. H. Rischke, PRD100, 056018 (2019)
  J.-H. Gao and Z.-T. Liang, PRD100, 056021(2019)
  K. Hattori, Y. Hidaka, and D.-L. Yang, PRD100, 096011 (2019)
  Z. Wang, X. Guo, S. Shi, and P. Zhuang, PRD 100 (2019) 014015
  Y.-C. Liu, K. Mameda, and X.-G. Huang (2020),2002.03753
- Decompose W in transport equation into generators of Clifford algebra:

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right).$$

- Insert into transport equation for Wigner function.
- Get system of 32 coupled (differential) equations.
- Equations for $\mathcal{F}$ (scalar, "particle distribution") and $\mathcal{S}_{\mu\nu}$ (tensor, "dipole moment") decouple from rest.
- Determine $\mathcal{V}_\mu$ ("vector current"), $\mathcal{A}_\mu$ ("polarization"), $\mathcal{P}$ from $\mathcal{S}_{\mu\nu}, \mathcal{F}$.
- Results will hold up to order $\mathcal{O}(\hbar)$.

Conventions

- Notation: $W = W^{(0)} + \hbar W^{(1)} + \mathcal{O}(\hbar^2)$.
- To simplify notation: only write positive-energy parts of solutions.
- Polarization direction $n^\mu(x,p)$: space-like unit vector parallel to axial-vector current.
- Spin quantization direction: unit vector, purely spatial in particle rest frame.
  Here: chosen to be identical to polarization direction.

$$\bar{u}_s(x,p)\gamma^\mu\gamma^5 u_s(x,p) = 2ms\, n^\mu(x,p)$$

$\rightarrow$ Distribution function diagonal in spin indices!

$$f_{rs} = f_s \delta_{rs}$$

- Dirac spinors space-time dependent.
- Spin quantization direction in rest-frame n* space-time and momentum dependent.

$$\bar{u}^\star_s(x,p)\boldsymbol{\gamma}\gamma^5 u^\star_s(x,p) = 2ms\,\boldsymbol{n}^\star(x,p)$$

Zeroth-order Wigner function

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- Direct calculation yields

$$\begin{aligned}
\mathcal{F}^{(0)}(x,p) &= m\,\delta(p^2-m^2)V^{(0)}(x,p),\\
\mathcal{A}^{(0)}_\mu(x,p) &= m\,n^{(0)}_\mu\,\delta(p^2-m^2)A^{(0)}(x,p),\\
\mathcal{P}^{(0)}(x,p) &= 0,\\
\mathcal{V}^{(0)}_\mu(x,p) &= p_\mu\,\delta(p^2-m^2)V^{(0)}(x,p),\\
\mathcal{S}^{(0)}_{\mu\nu}(x,p) &= m\,\Sigma^{(0)}_{\mu\nu}\,\delta(p^2-m^2)A^{(0)}(x,p),
\end{aligned}$$

with

$$\begin{aligned}
V^{(0)}(x,p) &\equiv \frac{2}{(2\pi\hbar)^3}\sum_s f^{(0)}_s(x,p),\\
A^{(0)}(x,p) &\equiv \frac{2}{(2\pi\hbar)^3}\sum_s s f^{(0)}_s(x,p),\\
\Sigma^{(0)}_{\mu\nu} &\equiv -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}p^\alpha n^{(0)\beta}.
\end{aligned}$$

- Solution fulfills zeroth-order transport equation for Wigner function.

Next-to-leading order from transport equation

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- Insert zeroth-order solution into first-order transport equation.
- Determine general form of $\mathcal{F}$ and $\mathcal{S}^{\mu\nu}$ up to order $\hbar$ from constraints.
- Generalized on-shell conditions:

$$
\begin{aligned}
(p^2 - m^2)\mathcal{F} &= \frac{1}{2}\hbar F^{\mu\nu}\mathcal{S}_{\mu\nu} + \mathcal{O}(\hbar^2), \\
(p^2 - m^2)\mathcal{S}_{\mu\nu} &= \hbar F_{\mu\nu}\mathcal{F} + \mathcal{O}(\hbar^2).
\end{aligned}
$$

- with additional constraint:

$$
p_\mu \mathcal{S}^{\mu\nu} = -\frac{\hbar}{2}\nabla^\nu \mathcal{F} + \mathcal{O}(\hbar^2).
$$

- $\mathcal{V}^\mu$, $\mathcal{A}^\mu$ and $\mathcal{P}$ only couple to $\mathcal{F}$ and $\mathcal{S}^{\mu\nu}$:

$$
\begin{aligned}
\mathcal{V}^\mu &= \frac{1}{m}\left(p^\mu \mathcal{F} - \frac{1}{2}\hbar \nabla_\nu \mathcal{S}^{\nu\mu}\right) + \mathcal{O}(\hbar^2), \\
\mathcal{A}^\mu &= -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu \mathcal{S}_{\alpha\beta} + \mathcal{O}(\hbar^2), \\
\mathcal{P} &= -\frac{1}{2m}\hbar \nabla_\mu \mathcal{A}^\mu + \mathcal{O}(\hbar^2).
\end{aligned}
$$

General results up to $\mathcal{O}(\hbar)$

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$$
\begin{aligned}
\mathcal{F} &= m\left[V\,\delta(p^2-m^2)-\frac{\hbar}{2}F^{\mu\nu}\Sigma^{(0)}_{\mu\nu}A^{(0)}\,\delta'(p^2-m^2)\right]+\mathcal{O}(\hbar^2)\,,\\
\mathcal{S}_{\mu\nu} &= m\left[\bar{\Sigma}_{\mu\nu}\delta(p^2-m^2)-\hbar F_{\mu\nu}V^{(0)}\delta'(p^2-m^2)\right]+\mathcal{O}(\hbar^2)\,,\\
\mathcal{P} &= \frac{\hbar}{4m}\epsilon^{\mu\nu\alpha\beta}\nabla_\mu\left[p_\nu\Sigma^{(0)}_{\alpha\beta}A^{(0)}\,\delta(p^2-m^2)\right]+\mathcal{O}(\hbar^2)\,,\\
\mathcal{V}_\mu &= \delta(p^2-m^2)\left[p_\mu V+\frac{\hbar}{2}\nabla^\nu\Sigma^{(0)}_{\mu\nu}A^{(0)}\right]\\
&\quad -\hbar\left[\frac{1}{2}p_\mu F^{\alpha\beta}\Sigma^{(0)}_{\alpha\beta}+\Sigma^{(0)}_{\mu\nu}F^{\nu\alpha}p_\alpha\right]A^{(0)}\,\delta'(p^2-m^2)+\mathcal{O}(\hbar^2)\,,\\
\mathcal{A}_\mu &= -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}p^\nu\bar{\Sigma}^{\alpha\beta}\,\delta(p^2-m^2)+\hbar\tilde{F}_{\mu\nu}p^\nu V^{(0)}\,\delta'(p^2-m^2)+\mathcal{O}(\hbar^2)\,,
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{\Sigma}^{(0)\mu\nu} &= \Sigma^{(0)\mu\nu}A^{(0)},\\
p_\nu\bar{\Sigma}^{\mu\nu} &= \hbar\nabla^\mu V^{(0)}.
\end{aligned}
$$

So what does this mean?

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- V and $\bar{\Sigma}^{\mu\nu}$ have to be determined through kinetic equations and constraint equation.
- By Taylor expansion of δ-function:

$$\mathcal{F} = \frac{2}{(2\pi\hbar)^3} m \sum_s \delta\left(p^2 - m^2 - \hbar \frac{s}{2} \Sigma^{(0)}_{\mu\nu} F^{\mu\nu}\right) f_s$$

  Modified on-shell condition.
  f_s distribution functions for spin-up ($s = +$) and spin-down ($s = -$) particles, $V = f_+ + f_-$ and $A^{(0)} = f^{(0)}_+ - f^{(0)}_-$.
- Write

$$\bar{\Sigma}^{\mu\nu} \equiv \Xi_{\mu\nu} + \frac{\hbar}{2} \chi_{\mu\nu}$$

  with

$$p_\nu \Xi^{\mu\nu} = 0,$$

  "classical" dipole moment;

$$p_\nu \chi^{\mu\nu} = \nabla^\mu V^{(0)},$$

  dipole moment induced by gradients.

Collisionless Boltzmann equation for massive spin-1/2 particles

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- Generalized Boltzmann equation

$$\sum_s \delta\left(p^2 - m^2\right)\left\{p^\mu \partial_{x\mu} f_s + \partial_{p\mu}\left[F^{\mu\nu} p_\nu + \frac{\hbar}{4} s \Sigma^{(0)\nu\rho}(\partial^\mu F_{\nu\rho})\right] f_s\right\} = 0$$

- Force on particle: first Mathisson-Papapetrou-Dixon (MPD) equation
  $\rightarrow$ Particle with classical dipole moment $\Sigma^{(0)\mu\nu}$ in electromagnetic field:

W. Israel, General Relativity and Gravitation 9 (1978) 451

$$m \frac{d}{d\tau} p^\mu = F^{\mu\nu} p_\nu + \frac{\hbar}{4} s \Sigma^{(0)\nu\rho}(\partial^\mu F_{\nu\rho}) .$$

τ: worldline parameter, $\frac{d}{d\tau} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{p}^\mu \frac{\partial}{\partial p^\mu}$.

Kinetic equation for dipole moment

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- $\bar{\Sigma}_{\mu\nu}$ determined by kinetic equation for dipole moment:

$$\delta(p^2 - m^2)\left[p \cdot \nabla \bar{\Sigma}^{\mu\nu} - \bar{\Sigma}^{\lambda\nu} F^{\mu}_{\ \lambda} + \bar{\Sigma}^{\lambda\mu} F^{\nu}_{\ \lambda} + \frac{1}{2}(\partial_{x\alpha} F^{\mu\nu})\partial_{p\alpha} V^{(0)}\right] = 0.$$

- To zeroth order:

$$m\frac{d}{d\tau}\Sigma^{(0)\mu\nu} = \Sigma^{(0)\lambda\nu} F^{\mu}_{\ \lambda} - \Sigma^{(0)\lambda\mu} F^{\nu}_{\ \lambda},.$$

- Recover second MPD equation for dipole-moment tensor $\Sigma^{(0)}_{\mu\nu}$!

W. Israel, General Relativity and Gravitation 9 (1978) 451

- Equivalent to Bargmann-Michel-Telegdi (BMT) equation

V. Bargmann, L. Michel, and V.L. Telegdi, PRL 2 (1959) 435

$$m\frac{d}{d\tau} n^{(0)\mu} = F^{\mu\nu} n^{(0)}_{\nu},$$

with classical spin vector

$$n^{(0)\mu} = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu \Sigma^{(0)}_{\alpha\beta}$$

Massless limit

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- Non-relativistic dipole-moment tensor connected to spin three-vector n^k:

$$\Sigma^{ij} = \epsilon^{ijk} n^k .$$

- For massive particles: define spin in rest frame.

U. Heinz, PLB 144 (1984) 228

$$\Sigma^{\mu\nu} = -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta .$$

- For massless particles: define spin in arbitrary frame with four-velocity u^μ. Spin vector n^μ is always parallel to momentum.

J-Y. Chen, D.T. Son, and M. Stephanov, PRL 115 (2015) 021601

$$\Sigma_u^{\mu\nu} = -\frac{1}{p \cdot u} \epsilon^{\mu\nu\alpha\beta} u_\alpha p_\beta .$$

- Massless limit: replace massive by massless dipole-moment tensor $\Sigma^{\mu\nu} \to \Sigma_u^{\mu\nu}$.
- Result agrees with previously known massless solution!

Y. Hidaka, S. Pu, and D-L. Yang, PRD 95 (2017) 091901

A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98 (2018) 036010

J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, PRD 98 (2018) 036019

Global equilibrium (standard collision term)

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- Assume standard (local) collision term which vanishes in global equilibrium
- Equilibrium distribution function:

$$f_s^{eq} = (e^{g_s} + 1)^{-1},$$

with g linear combination of conserved quantities charge, momentum, and angular momentum:

$$g_s = \beta U \cdot \pi - \beta \mu_s + \frac{\hbar}{4} s\, \omega_{\mu\nu} \Sigma^{\mu\nu}.$$

Here, $\pi_\mu \equiv p_\mu + A_\mu$ is canonical momentum, U_μ is fluid velocity, $\beta \equiv \frac{1}{T}$ is inverse temperature, and μ_s is chemical potential.

- Conditions for global equilibrium: Boltzmann equation has to be fulfilled.

$$\partial_\mu \mu_s = 0, \qquad \partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0,$$

$$\omega_{\mu\nu} = \frac{1}{2}(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu)$$

- Last condition only required in presence of electromagnetic field.

Vector current in global equilibrium

- Vector current is explicitly calculated as:

$$\mathcal{V}^\mu = \frac{2}{(2\pi\hbar)^3}\sum_s \Big[\delta(p^2-m^2)\left(p^\mu - m\hbar\frac{s}{2}\tilde{\omega}^{\mu\nu}n_\nu^{(0)}\partial_{\beta\cdot\pi}\right) + \hbar s\tilde{F}^{\mu\nu}n_\nu^{(0)}\delta'(p^2-m^2) + \hbar\frac{s}{2m}\delta(p^2-m^2)\epsilon^{\nu\mu\alpha\beta}p_\alpha\nabla_\nu n_\beta^{(0)}\Big] f_s^{(0)},$$

with zeroth-order equilibrium distribution function

$$f_s^{(0)} = [\exp(\beta U\cdot\pi - \beta\mu_s)+1]^{-1},$$

- Analogue of chiral vortical effect (CVE) for massive particles.

  D. T. Son and P. Surowka, PRL 103 (2009) 0906.5044

- Analogue of chiral magnetic effect (CME).

  D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, NPA 803 (2008) 0711.0950

- Dual thermal vorticity tensor: $\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}$.

Axial-vector current in global equilibrium

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- Obtain expression for axial-vector current:

$$\mathcal{A}^\mu = \frac{2}{(2\pi\hbar)^3}\sum_s \Bigg[\delta(p^2-m^2)\left(s\, m\, n^{(0)\mu} - \frac{\hbar}{2}\tilde{\omega}^{\mu\nu} p_\nu \partial_{\beta\cdot\pi}\right) + \hbar \tilde{F}^{\mu\nu} p_\nu \delta'(p^2-m^2)\Bigg] f_s^{(0)} - \frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta} p_\nu \Xi_{\alpha\beta}\, \delta(p^2-m^2)\,.$$

- Classical spin precession.
- Analogue of axial chiral vortical effect (ACVE).
- Analogue of chiral separation effect (CSE).

D. E. Kharzeev, J. Liao, S. A. Voloshin, and G. Wang, Prog. Part. NP 88 (2016), 1511.04050

Summary: free streaming

- Derived collisionless transport equation for distribution function and polarization for massive spin-1/2 particles in inhomogeneous electromagnetic fields.
- Recovered classical equations of motion.
- Solution agrees with previously known massless solution in massless limit.
- Derived explicit expressions for currents in global equilibrium under assumption of vanishing collision term.
- Spin potential $\omega^{\mu\nu}$ can be shown to be equal to thermal vorticity only in presence of electromagnetic fields.
- Is there something missing? $\rightarrow$ Collision term!

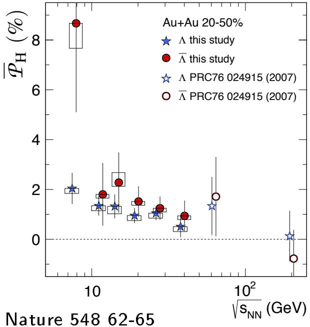
How to obtain spin polarization from vorticity?

Polarization from vorticity?

- Large global angular momentum created in noncentral heavy-ion collisions.
- Is orbital angular momentum converted into spin?
  Does this generate spin polarization in hot and dense matter?

Experimental observation - Λ polarization

- Measurement of Λ hyperon polarization along angular momentum direction



L. Adamczyk et al. (STAR), Nature 548 62-65

- Quark-gluon plasma is the "most vortical fluid ever observed"

$$\omega \approx (9 + 1) \times 10^{21} \mathrm{s}^{-1}$$

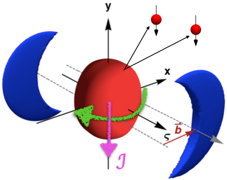
Great Red Spot of Jupiter $10^{-4}\,\mathrm{s}^{-1}$,
Turbulent flow superfluid He-II $150\,\mathrm{s}^{-1}$, Superfluid nanodroplets $10^{7}\,\mathrm{s}^{-1}$

Polarization from vorticity?

- Large global angular momentum created in noncentral heavy-ion collisions.
- Is orbital angular momentum converted into spin?
  Does this generate spin polarization in hot and dense matter?
  Yes!
- Connect spin polarization and vorticity!
- How to describe this in fluid dynamics?
- How to derive this from microscopic theory?

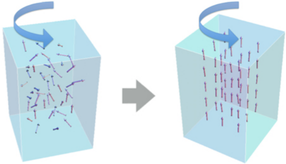
Rotation and polarization

- Heavy-ion collisions: Global rotation leads to polarization.



Picture by Radoslaw Ryblewski

- Condensed matter: **Barnett effect**



Picture by Mamoru Matsuo

Ferromagnet gets magnetized when it rotates

- What can we learn from the nonrelativistic case?

Micropolar fluids

G. Lukaszewicz, Micropolar Fluids, Theory and Applications (Birkhäuser Boston, 1999)

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- Simple model with many applications: spintronics, chiral active fluids,...
  R. Takahashi, M. Matsuo, M. Ono, K. Harii, H. Chudo, S. Okayasu, J. Ieda, S. Takahashi, S. Maekawa, and E. Saitoh, Nature Physics 12, 52 (2016)
  D. Banerjee, A. Souslov, A. G. Abanov, and V. Vitelli, Nature communications 8, 1 (2017)
- Fluid of rigid, randomly oriented particles with internal angular momentum $\boldsymbol{\ell}$.
- Mass density ρ, fluid velocity $\boldsymbol{u}$,
  non-symmetric stress tensor T^{ij}.
- Conservation of total angular momentum

$$\frac{d}{dt}\int_{\Omega(t)} d^3x\,\rho(\ell^i+\epsilon^{ijk}x^j u^k) = \int_{\partial\Omega(t)} d\Sigma^l (C^{li}+\epsilon^{ijk}x^j T^{lk})$$

  Change in volume element given by surface flow described by stress for momentum, "couple stress" for internal angular momentum.
- After short calculation:

$$\rho\left(\partial^0+u^j\partial^j\right)\ell^i = \partial^j C^{ji}+\epsilon^{ijk}T^{jk}$$

- Gain or loss of internal angular momentum: couple stress tensor and antisymmetric part of stress tensor!

Polarization from vorticity?

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- Is orbital angular momentum converted into spin?
  Does this generate spin polarization in hot and dense matter?
  Yes!
- Connect spin polarization and vorticity!
- How to describe this with fluid dynamics?
  - Antisymmetric part of energy-momentum tensor
  - Couple stress tensor
- How to derive this from microscopic theory?

Nonrelativistic kinetic theory

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- Nonrelativistic hydrodynamics with spin from kinetic theory was studied long time ago.

  S. Hess and L. Waldmann, Zeitschrift für Naturforschung A 26, 1057 (1971)
- Assumes local collision term.
- No orbital angular momentum in collision.
- Spin is conserved separately!
- Equilibrium polarization vanishes.

> "There is another effect which we cannot describe with a local collision operator (even in thermal equilibrium) : the orientation of the spin by a local or uniform rotation of the system (BARNETT effect)".
>
> S. Hess and L. Waldmann, Zeitschrift für Naturforschung A 26, 1057 (1971)

- Modify symmetric stress tensor by hand:
  "antisymmetric part of stress tensor $\propto$ spin polarization - vorticity."
- Phenomenological treatment of spin-vorticity term in equations of motion.

Polarization from vorticity?

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  - Antisymmetric part of energy-momentum tensor
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- How to derive this from microscopic theory?
  - Kinetic theory with nonlocal collisions
  - Equilibrium conditions?

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  - Calculate nonlocal collision term from quantum field theory.

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  - Calculate nonlocal collision term from quantum field theory.
  - Use Wigner function.

Including the collision term

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- Dirac equation with general interaction:

$$(i\hbar\gamma \cdot \partial - m)\,\psi = \hbar\rho$$

- Reminder: equation of motion for Wigner function
  S. R. De Groot, W. A. Van Leeuwen, and C. G. Van Weert, Relativistic Kinetic Theory. Principles and Applications (North-Holland, 1980)
  D. Vasak, M. Gyulassy, and H. T. Elze, AP 173, 462 (1987)

$$\left[\gamma \cdot \left(p + i\,\frac{\hbar}{2}\partial\right) - m\right] W = \hbar\,\mathcal{C}$$

collision term

$$\mathcal{C}_{\alpha\beta} \equiv \int \frac{d^4 y}{(2\pi\hbar)^4}\, e^{-\frac{i}{\hbar}p\cdot y} \left\langle : \bar{\psi}_\beta(x_1)\rho_\alpha(x_2) : \right\rangle \ .$$

- Now: Include nonlocal collision term $\mathcal{C}$,
  neglect electromagnetic mean fields.

  NW, E. Speranza, X.-l. Sheng, Q. Wang, and D.H. Rischke, arXiv:2005.01506 (2020)

- Idea: Expand Wigner function and collision term up to first order in gradients (equivalent to $\hbar$ expansion).

Component equations

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- Want to determine vector current $\mathcal{V}^\mu$ and axial-vector current $\mathcal{A}^\mu$ from equation of motion.
- Again obtain 32 coupled differential equations from Clifford decomposition.
- If spin effects are at least $\mathcal{O}(\hbar)$

$$\mathcal{V}^\mu = \frac{1}{m} p^\mu \bar{\mathcal{F}} + \mathcal{O}(\hbar^2)$$

where

$$\bar{\mathcal{F}} \equiv \mathcal{F} - \frac{\hbar}{m^2} p^\mu \mathrm{Re}\,\mathrm{Tr}(\gamma_\mu \mathcal{C})$$

- Relevant transport equations:

$$p \cdot \partial \bar{\mathcal{F}} = m\, C_F, \qquad p \cdot \partial \mathcal{A}^\mu = m\, C_A^\mu$$

with

$$C_F = 2 \mathrm{Im}\, \mathrm{Tr}(\mathcal{C}), \qquad C_A^\mu \equiv -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\nu \mathrm{Im}\, \mathrm{Tr}(\sigma_{\alpha\beta} \mathcal{C})$$

Spin in phase space

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- In order to account for spin dynamics enlarge phase space
  J. Zamanian, M. Marklund, and G. Brodin, NJP 12, 043019 (2010)
  W. Florkowski, R. Ryblewski, and A. Kumar, Prog. Part. Nucl. Phys. 108, 103709 (2019)
- Introduce new phase-space variable $\mathfrak{s}^\mu$

$$\mathfrak{f}(x,p,\mathfrak{s}) \equiv \frac{1}{2}\left[\bar{\mathcal{F}}(x,p) - \mathfrak{s}\cdot\mathcal{A}(x,p)\right] \; .$$

- Obtain $\bar{\mathcal{F}}$ and $\mathcal{A}^\mu$ via

$$\bar{\mathcal{F}} = \int dS(p)\, \mathfrak{f}(x,p,\mathfrak{s}) \; , \quad \mathcal{A}^\mu = \int dS(p)\, \mathfrak{s}^\mu \mathfrak{f}(x,p,\mathfrak{s})$$

with $dS(p) \equiv \frac{\sqrt{p^2}}{\sqrt{3}\pi} d^4\mathfrak{s}\, \delta(\mathfrak{s}^2+3)\delta(p\cdot\mathfrak{s})$.

- Boltzmann equation

$$p\cdot\partial\, \mathfrak{f}(x,p,\mathfrak{s}) = m\,\mathfrak{C}[\mathfrak{f}] \; ,$$

$$\mathfrak{C}[\mathfrak{f}] \equiv \frac{1}{2}(C_F - \mathfrak{s}\cdot C_A).$$

- All dynamics in one scalar equation!

Nonlocal collisions

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- Want to obtain collision term up to first order in gradients

$$\mathfrak{C}[\mathfrak{f}] = \mathfrak{C}_l[\mathfrak{f}] + \hbar\, \mathfrak{C}_{nl}[\mathfrak{f}] \,.$$

Local contribution + **Nonlocal** contribution

- Starting point:
  S. R. De Groot, W. A. Van Leeuwen, and C. G. Van Weert, Relativistic Kinetic Theory. Principles and Applications (North-Holland, 1980)

$$p \cdot \partial W = C$$

with

$$C_{\alpha\beta} = \frac{i}{2}\int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \Big\langle [\bar\rho(x_1)(-i\hbar\gamma\cdot\overleftarrow{\partial} + m)]_\beta \psi_\alpha(x_2) - \bar\psi_\beta(x_1)[(i\hbar\gamma\cdot\partial + m)\rho(x_2)]_\alpha \Big\rangle \,.$$

Calculation of collision term C

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- Expand ensemble average in initial n-particle scattering states.
- Neglect initial correlations (molecular chaos in infinite past).
- Assume binary scattering ($n = 2$).
- Low-density approximation:
  Identify initial Wigner function in collision term with interacting Wigner function W.

$$C_{\alpha\beta} = \frac{(2\pi\hbar)^6}{2(4m^4)} \sum_{r_1,r_2,s_1,s_2} \int d^4p_1 d^4p_2 d^4u_1 d^4u_2$$

$$\times\, {}_{\text{in}}\langle p_1 - \frac{1}{2}u_1, p_2 - \frac{1}{2}u_2; r_1, r_2 | \Phi_{\alpha\beta}(p) | p_1 + \frac{1}{2}u_2, p_2 + \frac{1}{2}u_2; s_1, s_2 \rangle_{\text{in}}$$

$$\times \prod_{j=1}^{2} \bar{u}_{s_j}(p_j + \frac{1}{2}u_j) \left[W(x, p_j)\delta^{(4)}(u_j) - i\hbar(\partial^\mu_{u_j}\delta^{(4)}(u_j))\partial_{x\mu} W(x, p_j) \right] u_{r_j}(p_j - \frac{1}{2}u_j)$$

- Consider contribution from zeroth and first order in gradients.
- Note: constant spin quantization direction, spinors $u_s(p)$ independent of space-time.

Nonlocal collision term: result

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- Long calculation → intuitive result:

$$\mathfrak{C}[\mathfrak{f}] = \int d\Gamma_1 d\Gamma_2 d\Gamma' \, \mathcal{W} \, [\mathfrak{f}(x + \Delta_1, p_1, \mathfrak{s}_1) \times \mathfrak{f}(x + \Delta_2, p_2, \mathfrak{s}_2) - \mathfrak{f}(x + \Delta, p, \mathfrak{s}) \mathfrak{f}(x + \Delta', p', \mathfrak{s}')] + \int d\Gamma_2 \, dS_1(p) \, \mathfrak{W} \, \mathfrak{f}(x + \Delta_1, p, \mathfrak{s}_1) \mathfrak{f}(x + \Delta_2, p_2, \mathfrak{s}_2)$$

$d\Gamma \equiv d^4 p \, dS(p)$

- Structure: Momentum and spin exchange + Spin exchange only
- Collision nonlocal, particle positions displaced by

$$\Delta^\mu = -\frac{\hbar}{2m(p \cdot \hat{t} + m)} \epsilon^{\mu\nu\alpha\beta} p_\nu \hat{t}_\alpha \mathfrak{s}_\beta$$

with $\hat{t} = (1, 0)$.

- $\mathcal{W}$, $\mathfrak{W}$ transition probabilities, depend on phase-space spins.
- Neglected momentum derivatives of scattering amplitudes.

Equilibrium (nonlocal collisions) I

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- Equilibrium condition: Collision term has to vanish.
- Ansatz for distribution function
  F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, AP. 338, 32 (2013)
  W. Florkowski, R. Ryblewski, and A. Kumar, Prog. Part. Nucl. Phys. 108, 103709 (2019)

$$\mathfrak{f}_{eq}(x,p,\mathfrak{s}) = \frac{m}{(2\pi\hbar)^3} \exp\left[-\beta(x)\cdot p + \frac{\hbar}{4}\Omega_{\mu\nu}(x)\Sigma_{\mathfrak{s}}^{\mu\nu}\right]\delta(p^2 - M^2)$$

  - β^μ - Lagrange multiplier for 4-momentum conservation
  - Spin potential $\Omega^{\mu\nu}$ - Lagrange multiplier for **total** angular momentum conservation
  - M - mass possibly modified by interactions
  - Dipole-moment tensor

$$\Sigma_{\mathfrak{s}}^{\mu\nu} \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha \mathfrak{s}_\beta$$

- Insert into $\mathfrak{C}[\mathfrak{f}]$ and expand up to first order in $\hbar$.
  $\Longrightarrow$ Zeroth-order collision term vanishes due to momentum conservation

Equilibrium (nonlocal collisions) II

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At first order in $\hbar$:

$$\mathfrak{C}[\mathfrak{f}_{eq}] = -\int d\Gamma' d\Gamma_1 d\Gamma_2\, \widetilde{\mathcal{W}}\, e^{-\beta\cdot(p_1+p_2)}$$
$$\times\left[\partial_\mu\beta_\nu\left(\Delta_1^\mu p_1^\nu + \Delta_2^\mu p_2^\nu - \Delta^\mu p^\nu - \Delta'^\mu p'^\nu\right) - \frac{1}{2}\Omega_{\mu\nu}\frac{\hbar}{2}\left(\Sigma_{\mathfrak{s}_1}^{\mu\nu} + \Sigma_{\mathfrak{s}_2}^{\mu\nu} - \Sigma_{\mathfrak{s}}^{\mu\nu} - \Sigma_{\mathfrak{s}'}^{\mu\nu}\right)\right]$$
$$-\int d\Gamma_2\, dS_1(p) dS'(p_2)\, \mathfrak{W}\, e^{-\beta\cdot(p+p_2)}$$
$$\times\left\{\partial_\mu\beta_\nu\left[(\Delta_1^\mu - \Delta^\mu)p^\nu + (\Delta_2^\mu - \Delta'^\mu)p_2^\nu\right] - \frac{1}{2}\Omega_{\mu\nu}\frac{\hbar}{2}(\Sigma_{\mathfrak{s}_1}^{\mu\nu} + \Sigma_{\mathfrak{s}_2}^{\mu\nu} - \Sigma_{\mathfrak{s}}^{\mu\nu} - \Sigma_{\mathfrak{s}'}^{\mu\nu})\right\} .$$

- Conservation of total angular momentum (orbital+spin) in a collision

$$J^{\mu\nu} = \Delta^\mu p^\nu - \Delta^\nu p^\mu + \frac{\hbar}{2}\Sigma_{\mathfrak{s}}^{\mu\nu}$$

- Conditions for vanishing of collision term at first order:

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0$$
$$\Omega_{\mu\nu} = \varpi_{\mu\nu} \equiv -\frac{1}{2}\partial_{[\mu}\beta_{\nu]} = \text{const.}$$
$$a_{[\mu}b_{\nu]} \equiv a_\mu b_\nu - a_\nu b_\mu$$

Equilibrium (nonlocal collisions) III

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Discussion

- Collision term vanishes under conditions for global equilibrium!
- No (standard) local equilibrium with nonlocal collisions.
- Confirm known result from statistical quantum field theory:
  In equilibrium spin potential equal to thermal vorticity.
  F. Becattini, PRL 108, 244502 (2012)
- Interpretation: When approaching equilibrium, non-vanishing vorticity converts orbital angular momentum into spin through nonlocal collisions
  $\Longrightarrow$Initially unpolarized fluid gets polarized, Barnett effect!

Polarization from vorticity?

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- Large global angular momentum created in noncentral heavy-ion collisions.
- Is orbital angular momentum converted into spin?
  Does this generate spin polarization in hot and dense matter?
  Yes!
- Connect spin polarization and vorticity!
- How to describe this with fluid dynamics?
  - Antisymmetric part of energy-momentum tensor
  - Couple stress tensor
- How to derive this from microscopic theory?
  - Kinetic theory with nonlocal collisions
  - Calculate nonlocal collision term from quantum field theory.
  - Equilibrium conditions

Spin hydrodynamics

- Conservation of total angular momentum tensor

$$J^{\lambda,\mu\nu} \equiv x^{\mu} T^{\lambda\nu} - x^{\nu} T^{\lambda\mu} + \hbar S^{\lambda,\mu\nu}$$

Energy-momentum tensor $T^{\mu\nu}$
Additional dynamical tensor: Spin tensor $S^{\lambda,\mu\nu}$

W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, PRC 97, no. 4, 041901 (2018)
W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza, PRD 97, no. 11, 116017 (2018)
W. Florkowski, F. Becattini, and E. Speranza, APB 49, 1409 (2018)
W. Florkowski, F. Becattini, and E. Speranza, PLB 789, 419 (2019)
K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, PLB795, 100 (2019)

- Equations of motion:

$$\partial_{\mu} T^{\mu\nu} = 0 \qquad \hbar\partial_{\lambda} S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- Definition of $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$ depends on choice of pseudo-gauge.

F. W. Hehl, Rept. Math. Phys. 9, 55 (1976)
E. Leader and C. Lorce, Phys. Rept. 541, 163 (2014)
F. Becattini, W. Florkowski, and E. Speranza, PLB789, 419 (2019)
E. Speranza, NW, arXiv:2007.00138 (2020)
L. Tinti, W. Florkowski, arXiv: 2007.04029 (2020)

Pseudo-gauge transformations

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- Pseudo-gauge transformation:

  F. W. Hehl, Rept. Math. Phys. 9, 55 (1976)

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{\hbar}{2}\partial_\lambda(\Phi^{\lambda,\mu\nu} + \Phi^{\nu,\mu\lambda} + \Phi^{\mu,\nu\lambda}),$$
$$S'^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_\rho Z^{\mu\nu,\lambda\rho}.$$

- Equations of motion invariant:

$$\partial_\mu T'^{\mu\nu} = 0 \qquad \hbar\partial_\lambda S'^{\lambda,\mu\nu} = T'^{\nu\mu} - T'^{\mu\nu}$$

- Total charges invariant

$$P^\nu \equiv \int d\Sigma_\mu T^{\mu\nu} = \int d\Sigma_\mu T'^{\mu\nu}$$
$$J^{\mu\nu} \equiv \int d\Sigma_\lambda J^{\lambda,\mu\nu} = \int d\Sigma_\lambda J'^{\lambda,\mu\nu}$$

- Hydrodynamics: Densities $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$ are dynamical variables
  $\Longrightarrow$ Pseudo-gauge choice expected to be important!

Canonical pseudo-gauge

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- Apply Noether's theorem to Dirac Lagrangian

$$T_C^{\mu\nu} = \int d^4p\, p^\nu \mathcal{V}^\mu = \int d\Gamma\, p^\mu p^\nu \mathfrak{f} + \mathcal{O}(\hbar^2),$$

$$S_C^{\lambda,\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho} \int d^4p\, \mathcal{A}_\rho = -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho} \int d\Gamma\, \mathfrak{s}_\rho \mathfrak{f}$$

- Equations of motion up to first order

$$\partial_\mu T_C^{\mu\nu} = \int d\Gamma\, p^\nu\, \mathfrak{C}[\mathfrak{f}] = 0,$$

Momentum: collisional invariant

$$\hbar\, \partial_\lambda S_C^{\lambda,\mu\nu} = \int d\Gamma\, \frac{\hbar}{2} \left(\Sigma_\mathfrak{s}^{\mu\nu}\, \mathfrak{C}[\mathfrak{f}] + p^{[\mu} \Sigma_\mathfrak{s}^{\nu]\lambda} \partial_\lambda \mathfrak{f}(x,p,\mathfrak{s}) \right) = T_C^{[\nu\mu]}$$

Spin: collisional invariant only for local collisions
Antisymmetric part of energy-momentum tensor:
nonzero even in absence of interactions and in global equilibrium!

- Not consistent with physical picture: Conversion between spin and orbital angular momentum only through interactions until global equilibrium is reached.

Belinfante pseudo-gauge

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- Choose

$$\Phi^{\lambda,\mu\nu} = S_C^{\lambda,\mu\nu}$$

- Resulting set of tensors:

$$\begin{aligned} T_B^{\mu\nu} &= \frac{1}{2}\int d^4p\,(p^\nu \mathcal{V}^\mu + p^\mu \mathcal{V}^\nu) = \int d\Gamma\, p^\mu p^\nu \mathfrak{f} + \mathcal{O}(\hbar^2), \\ S_B^{\lambda,\mu\nu} &= 0 \end{aligned}$$

- Energy-momentum tensor symmetric.
- But: Spin tensor vanishes.
- Spin degrees of freedom cannot be described through energy-momentum tensor only.
- Is there a possibility to have a symmetric energy-momentum tensor, but nonzero spin tensor?

HW pseudo-gauge

- Idea for free fields:
  Apply Noether's theorem to Klein-Gordon Lagrangian for spinors
  J. Hilgevoord and S. Wouthuysen, Nuclear Physics 40, 1 (1963)

$$\mathcal{L}_{KG} = \frac{1}{2m}(\hbar^2 \partial_\mu \bar{\psi} \partial^\mu \psi - m^2 \bar{\psi}\psi)$$

- Result:

$$T^{\mu\nu}_{HW} = \frac{1}{m} \int d^4p \left[p^\mu p^\nu + \frac{\hbar^2}{4} \partial^\mu \partial^\nu - \frac{\hbar^2}{4} g^{\mu\nu} \partial^2 \right] \mathcal{F}$$

$$S^{\lambda,\mu\nu}_{HW} = \frac{1}{2m} \int d^4p \, p^\lambda \mathcal{S}^{\mu\nu}$$

- Energy-momentum tensor symmetric for free fields.
- Conserved (nonzero) spin tensor.
- Physical interpretation?

Pseudo-gauge and frame choice I

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- Nonrelativistic spin operator given by Pauli matrices: $\frac{1}{2}\boldsymbol{\sigma}$
- How to generalize to relativistic theory?
- Spin vector $\boldsymbol{S}$ connected to global spin by

$$S^{ij} = \epsilon^{ijk} S^k .$$

  Obviously no Lorentz tensor.
- Make this covariant:

$$S_n^{\mu\nu} = -\epsilon^{\mu\nu\alpha\beta} n_\alpha S_\beta$$

  Spin defined in the frame moving with four-velocity $n^\mu \Longleftrightarrow n_\mu S_n^{\mu\nu} = 0$.
- Different choices of pseudo-gauge: different choices of frame vector.
  M. H. L. Pryce, Proc. Roy. Soc. Lond., A195:62–81, 1948
  C. Lorcé, Eur. Phys. J. C (2018) 78:785
  E. Speranza, NW, arXiv:2007.00138 (2020)
- One preferred reference frame for massive particles: rest frame.

Pseudo-gauge and frame choice II

Global spin from spin tensor:

$$S^{\mu\nu} \equiv \int d^3x \, S^{0,\mu\nu}$$

- Canonical choice:
  $\Longrightarrow$ Spin tensor not conserved for free fields
  $\Longrightarrow$ Global spin no Lorentz tensor
  $\Longrightarrow$ Equal to nonrelativistic spin in any frame,

$$S_C^{0\nu} = 0, \qquad n_C^\mu = (1,0,0,0).$$

- HW choice:
  $\Longrightarrow$ Spin tensor conserved for free fields
  $\Longrightarrow$ Global spin is Lorentz tensor
  $\Longrightarrow$ Equal to nonrelativistic spin in rest frame,

$$P_\mu S_{HW}^{\mu\nu} = 0, \qquad n_{HW}^\mu = \frac{1}{m} P^\mu .$$

HW pseudo-gauge for interacting case

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- Interacting case:

  NW, E. Speranza, X.-l. Sheng, Q. Wang, and D.H. Rischke, arXiv:2005.01506 (2020)

  Obtain energy-momentum and spin tensor by pseudo-gauge transformation from canonical tensors.
- Choice of $\Phi^{\lambda,\mu\nu}$:
  - Recover HW tensors for zero interactions.
  - Obtain physically meaningful equations of motion (see next slide).
- Result:

$$T^{\mu\nu}_{HW} = \int d\Gamma p^\mu p^\nu \mathfrak{f}(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2) ,$$

$$S^{\lambda,\mu\nu}_{HW} = \int d\Gamma p^\lambda \left(\frac{1}{2}\Sigma^{\mu\nu}_{\mathfrak{s}} - \frac{\hbar}{4m^2} p^{[\mu}\partial^{\nu]} \right) \mathfrak{f}(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2) .$$

Equations of motion with collsions

- Using Boltzmann equation

$$\partial_\mu T^{\mu\nu}_{HW} = \int d\Gamma\, p^\nu\, \mathfrak{C}[\mathfrak{f}] = 0 \,,$$
$$\hbar\, \partial_\lambda S^{\lambda,\mu\nu}_{HW} = \int d\Gamma\, \frac{\hbar}{2} \Sigma^{\mu\nu}_{\mathfrak{s}}\, \mathfrak{C}[\mathfrak{f}] = T^{[\nu\mu]}_{HW} \,.$$

- Energy-momentum conserved in a collision
- Spin not conserved in nonlocal collisions $\Leftrightarrow T^{[\nu\mu]}_{\rm HW} \neq 0$
  $\Rightarrow$ Conversion between spin and orbital angular momentum
- $T^{[\nu\mu]}_{\rm HW} = 0$
  (i) for local collisions, as spin is collisional invariant
  (ii) in global equilibrium, as collision term vanishes
- With nonlocal collisions out of global equilibrium: dynamics dissipative

Polarization from vorticity?

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- Large global angular momentum created in noncentral heavy-ion collisions.
- Is orbital angular momentum converted into spin?
  Does this generate spin polarization in hot and dense matter?
  Yes!
- Connect spin polarization and vorticity!
- How to describe this with fluid dynamics?
  - Antisymmetric part of energy-momentum tensor
  - Couple stress tensor
    $\Longrightarrow$ Nonrelativistic limit!
- How to derive this from microscopic theory?
  - Kinetic theory with nonlocal collisions
  - Calculate nonlocal collision term from quantum field theory.
  - Equilibrium conditions

Nonrelativistic limit

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- $p^\mu \to m(1, \boldsymbol{v})$, $\Sigma_{\mathfrak{s}}^{\mu\nu} \to \epsilon^{ijk}\mathfrak{s}^k$

$$T_{HW}^{[ji]} = m\epsilon^{ijk}\partial^0 \left\langle \frac{\hbar}{2}\mathfrak{s}^k \right\rangle + m\epsilon^{ijk}\partial^l \left\langle v^l \frac{\hbar}{2}\mathfrak{s}^k \right\rangle$$

with $\langle \dots \rangle \equiv (m^2/2\pi\sqrt{3}) \int d^3v\, d^3\mathfrak{s}\, \delta(\mathfrak{s}^2 - 3)\,(\dots) f$

- Agreement with phenomenological result of nonrelativistic kinetic theory.

S. Hess and L. Waldmann, Zeitschrift für Naturforschung A 26, 1057 (1971)

- Comparison with micropolar fluids

G. Lukaszewicz, Micropolar Fluids, Theory and Applications (Birkhäuser Boston, 1999)

$$\rho \left(\partial^0 + u^j \partial^j\right) \ell^i = \partial^j C^{ji} + \epsilon^{ijk} T^{jk}$$

$\Longrightarrow$ Internal angular momentum

$$\rho\, \ell^i = m \left\langle \frac{\hbar}{2}\mathfrak{s}^i \right\rangle ,$$

$\Longrightarrow$ Couple stress tensor

$$C^{ji} = -\left\langle \frac{\hbar}{2}\mathfrak{s}^i p^j \right\rangle + m \left\langle \frac{\hbar}{2}\mathfrak{s}^i \right\rangle u^j .$$

Conclusions and outlook

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- Derivation of nonlocal collisions from quantum field theory
  - Collision term vanishes only in global equilibrium
  - Local collisions $\Longrightarrow$ Ideal spin hydrodynamics possible
    Nonlocal collisions $\Longrightarrow$ Always dissipative dynamics
- Spin hydrodynamics with HW pseudo-gauge
  - Antisymmetric part of energy-momentum tensor
    $\Longrightarrow$ Conversion between spin and orbital angular momentum
    $\Longrightarrow$ Vanishes with local collisions or in global equilibrium
- Nonrelativistic limit
  - Agreement with kinetic-theory result
  - Related HW energy-momentum tensor to stress tensor of micropolar fluids.
  - Found microscopic expression for couple stress tensor

Outlook: Derive second-order dissipative hydrodynamics with spin using method of moments.

G.S. Denicol, H. Niemi, E. Molnar, D.H. Rischke, PRD 85 (2012) 114047

Back-up

Collisonless kinetic equations

- Kinetic equations for V and $\bar{\Sigma}_{\mu\nu}$:

$$\begin{aligned} 0 &= \delta(p^2 - m^2)\left[p\cdot\nabla V + \frac{\hbar}{4}(\partial_x^\alpha F^{\mu\nu})\partial_{p\alpha}\bar{\Sigma}_{\mu\nu}\right] \\ &\quad -\frac{\hbar}{2}\delta'(p^2 - m^2)F^{\alpha\beta}\, p\cdot\nabla\bar{\Sigma}_{\alpha\beta} + \mathcal{O}(\hbar^2), \\ 0 &= \delta(p^2 - m^2)\left[p\cdot\nabla\bar{\Sigma}_{\mu\nu} - F^{\alpha}_{[\mu}\bar{\Sigma}_{\nu]\alpha} + \frac{\hbar}{2}(\partial_{x\alpha}F_{\mu\nu})\partial_p^\alpha V\right] \\ &\quad -\hbar\delta'(p^2 - m^2)F_{\mu\nu}\, p\cdot\nabla V + \mathcal{O}(\hbar^2). \end{aligned}$$

- Can we get rid of "δ'-terms"?

$$\hbar\,\delta(p^2 - m^2)p\cdot\nabla V \quad \in \quad \mathcal{O}(\hbar^2),$$

Omitting off-shell term

- Wigner function and kinetic equations invariant under transformation:

$$\begin{aligned} V &\rightarrow \hat{V} = V + (p^2 - m^2)\delta V, \\ \bar{\Sigma}_{\mu\nu} &\rightarrow \hat{\bar{\Sigma}}_{\mu\nu} = \bar{\Sigma}_{\mu\nu} - \hbar F_{\mu\nu}\delta V. \end{aligned}$$

- Find transformation such that

$$\int dp^0\, \delta'(p^2 - m^2) G(x,p)\, p \cdot \nabla \hat{V} \in \mathcal{O}(\hbar)$$

  for arbitrary $G(x,p)$.
- Analogously for $p \cdot \nabla \bar{\Sigma}_{\alpha\beta}$.
- Drop "δ'-terms" in kinetic equations without loss of generality!

Power counting

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- Our $\hbar$-expansion is equivalent to gradient expansion.
- We treat all gradients on the same level, i.e.
  - gradients in formal $\hbar$-expansion of Wigner function

$$W = W^{(0)} + \hbar W^{(1)} + \mathcal{O}(\hbar^2),$$

  - gradients in nonlocal expansion of collision term

$$\mathfrak{C} = \mathfrak{C}_l + \hbar \mathfrak{C}_{nl} + \mathcal{O}(\hbar^2),$$

  - and gradients in expansion of distribution function around equilibrium

$$f = f_{eq} + \delta f$$

  are considered to be of same order.
- This implies that $f^{(0)}$ contains only equilibrium contributions.
- $f^{(1)}$ contains equilibrium and off-equilibrium contributions.
- $\mathfrak{C}_{nl}$ is a functional only of $f^{(0)}$,
  $\mathfrak{C}_{nl}[f^{(1)}]$ would enter collision term at second order.