27 October 2020

How can relativity make Navier-Stokes unstable?

Lorenzo Gavassino

Nicolaus Copernicus Astronomical Center of the Polish Academy of Sciences

Based on

10.1103/PhysRevD.102.043018

In collaboration with
Dr. Marco Antonelli
Prof. Brynmor Haskell

CAMK
PAN

Diffusion equation

Standard Newtonian equation of the evolution of a temperature profile over time (1D case)

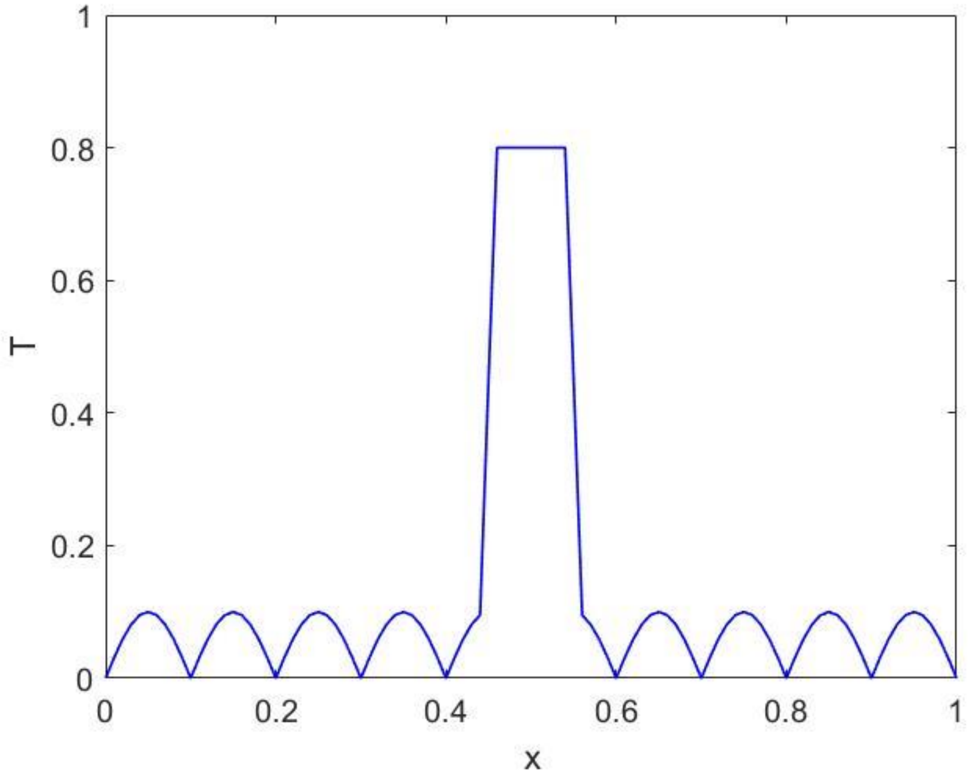
$$\frac{\partial T}{\partial t} = D \, \frac{\partial^2 T}{\partial \, x^2}$$

Diffusion equation

Standard Newtonian equation of the evolution of a temperature profile over time (1D case)

$$\frac{\partial T}{\partial t} = D \, \frac{\partial^2 T}{\partial \, x^2}$$

Initial profile $(t = 0)$

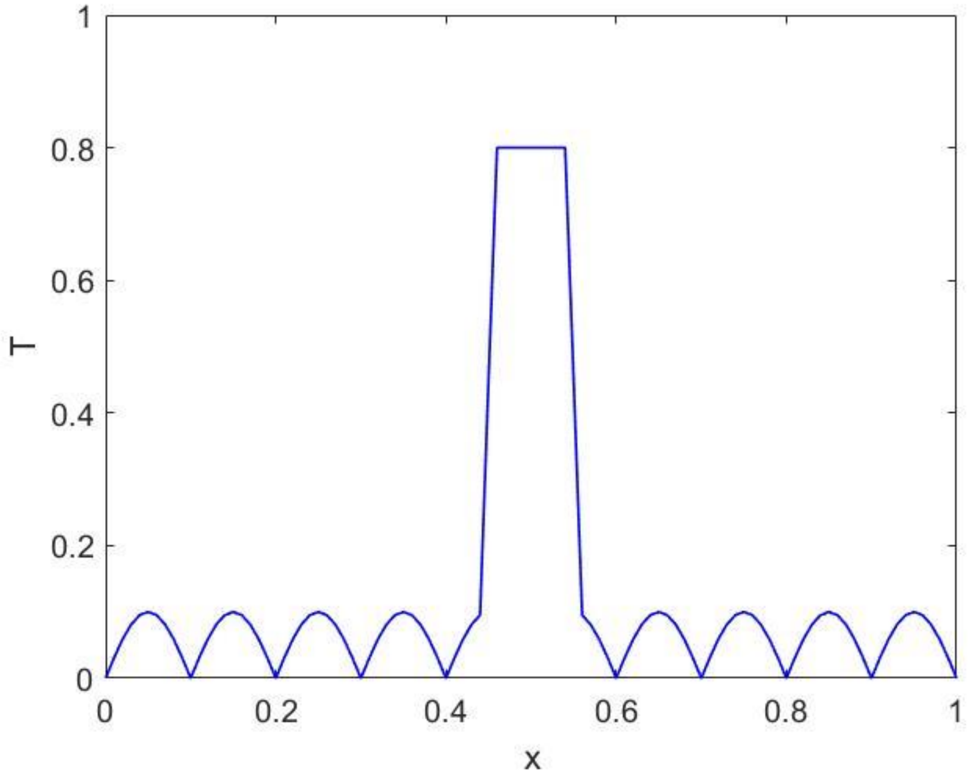


Diffusion equation

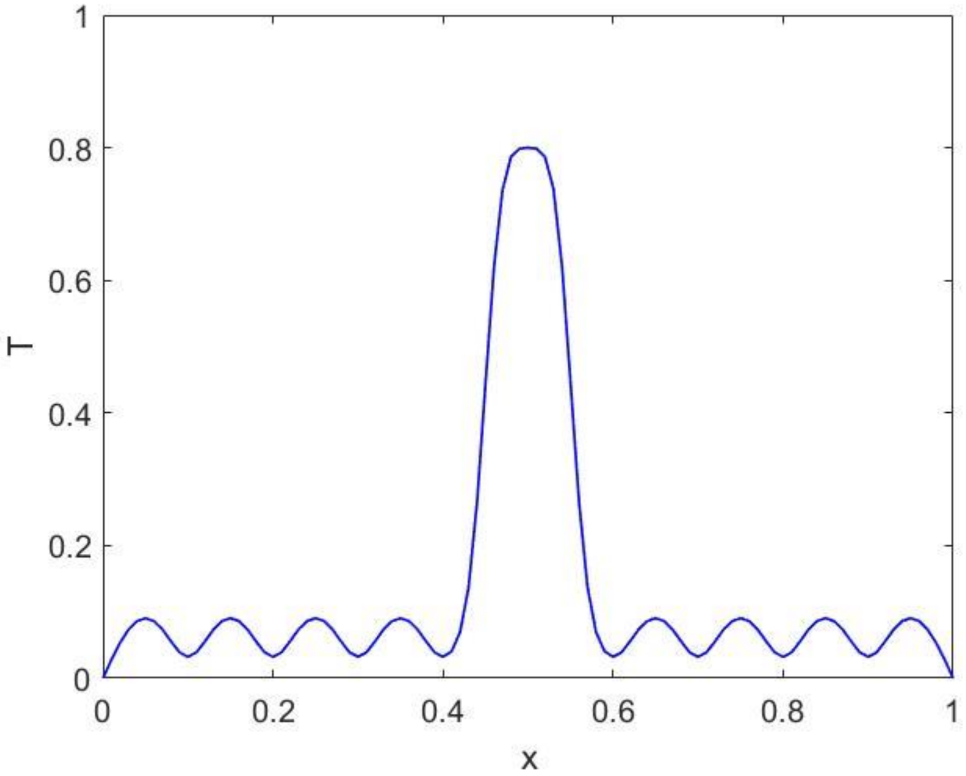
Standard Newtonian equation of the evolution of a temperature profile over time (1D case)

$$\frac{\partial T}{\partial t} = D\ \frac{\partial^2 T}{\partial\ x^2}$$

Initial profile $(t = 0)$



$t = 5$

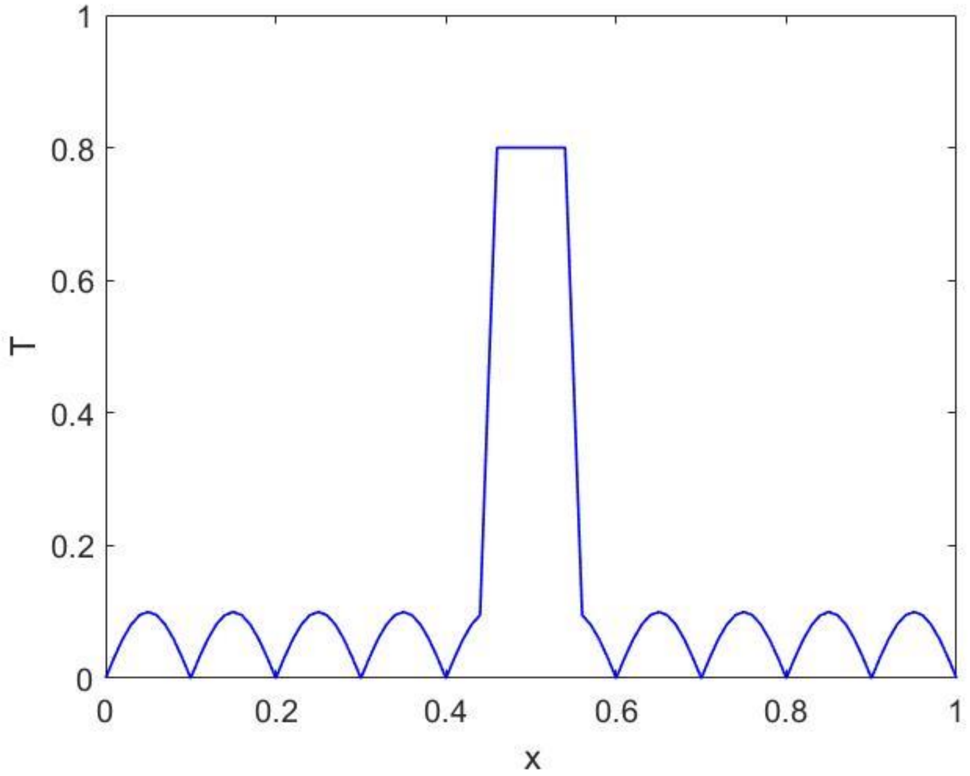


Diffusion equation

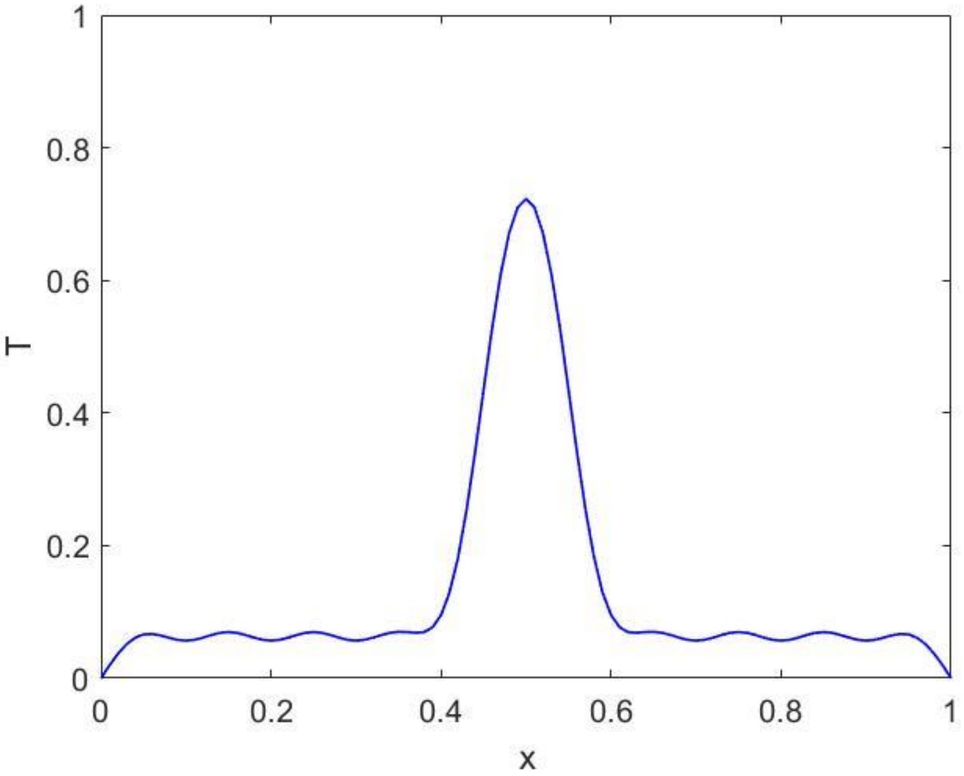
Standard Newtonian equation of the evolution of a temperature profile over time (1D case)

$$\frac{\partial T}{\partial t} = D\,\frac{\partial^2 T}{\partial\, x^2}$$

Initial profile $(t = 0)$



$t = 20$

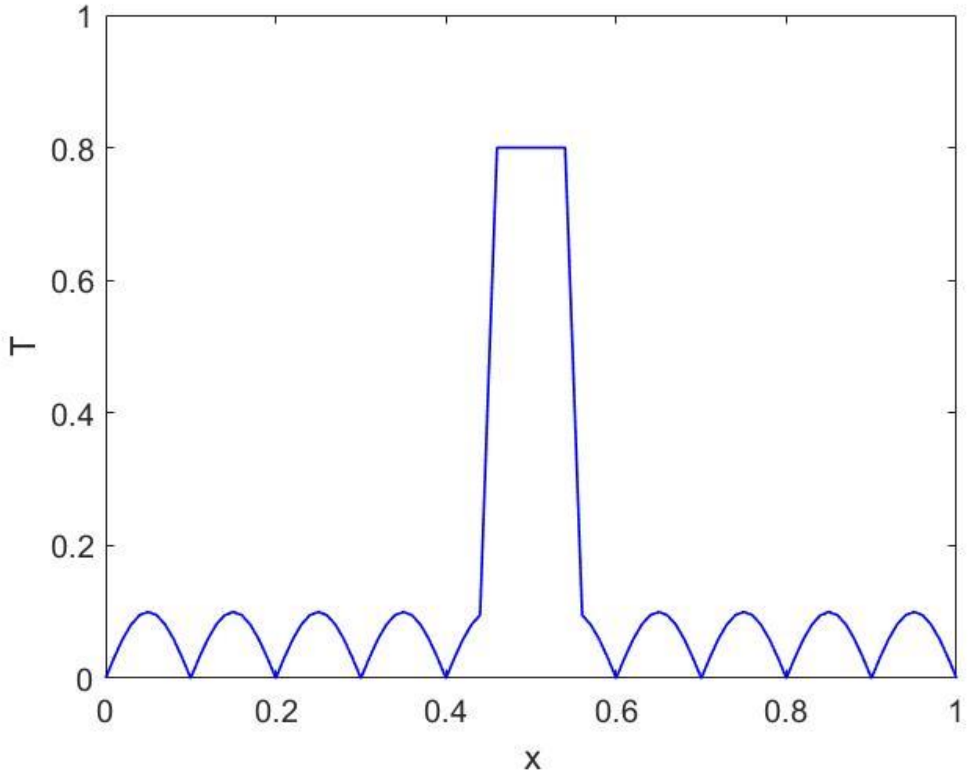


Diffusion equation

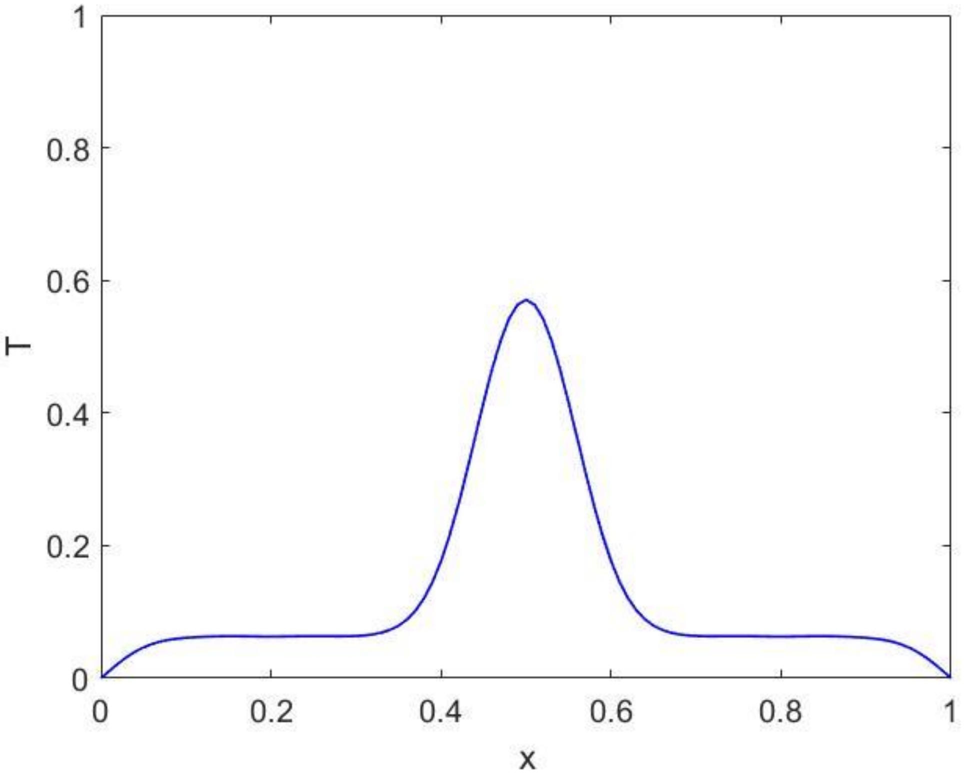
Standard Newtonian equation of the evolution of a temperature profile over time (1D case)

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$$

Initial profile ($t = 0$)



$t = 50$

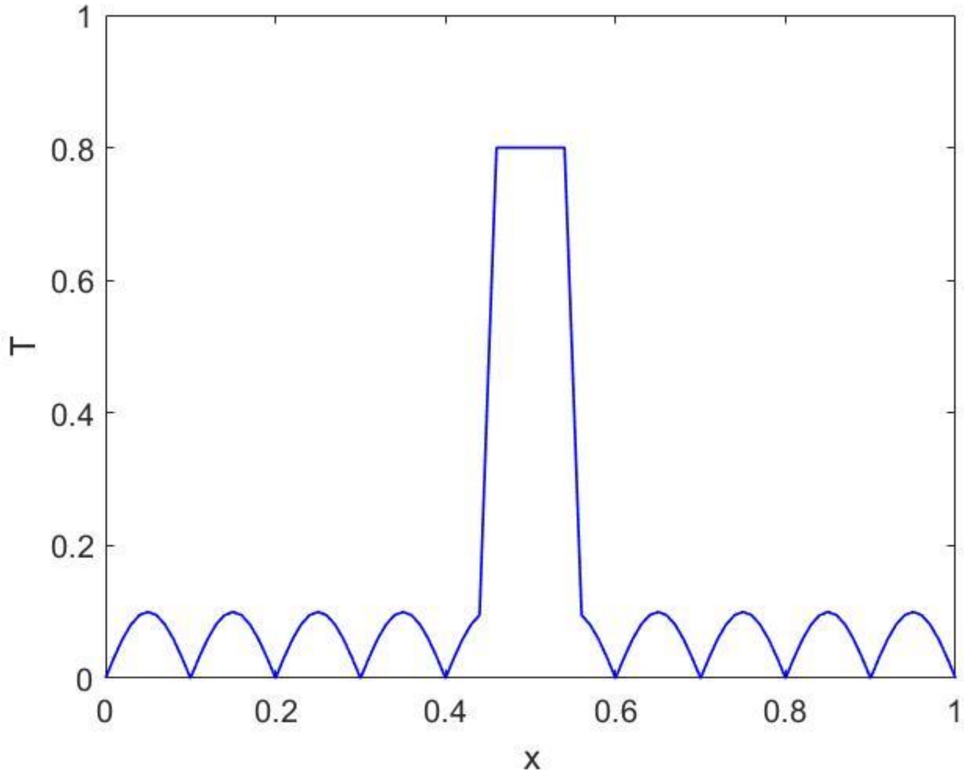


Diffusion equation

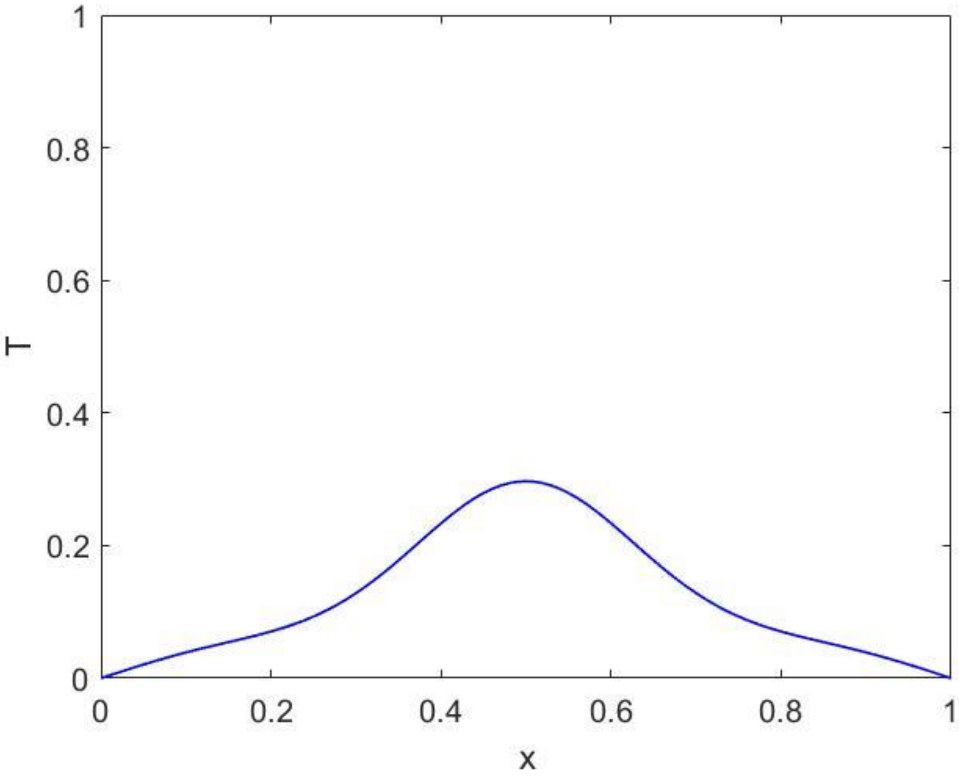
Standard Newtonian equation of the evolution of a temperature profile over time (1D case)

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$$

Initial profile $(t = 0)$



$t = 300$

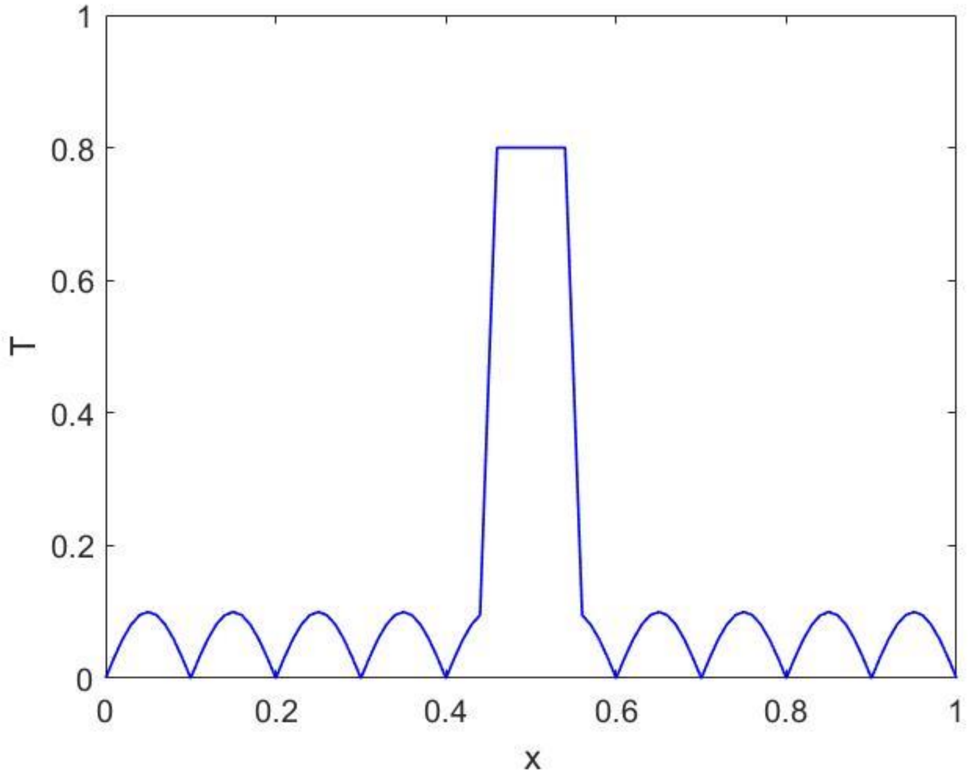


Diffusion equation

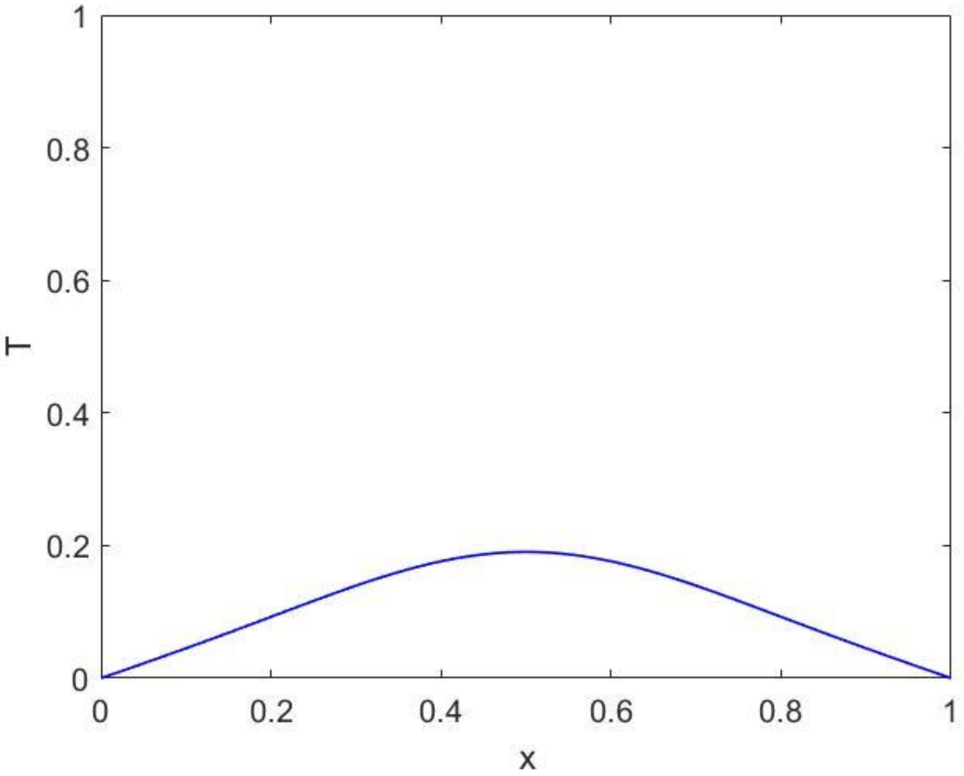
Standard Newtonian equation of the evolution of a temperature profile over time (1D case)

$$\frac{\partial T}{\partial t} = D\,\frac{\partial^2 T}{\partial\, x^2}$$

Initial profile $(t = 0)$



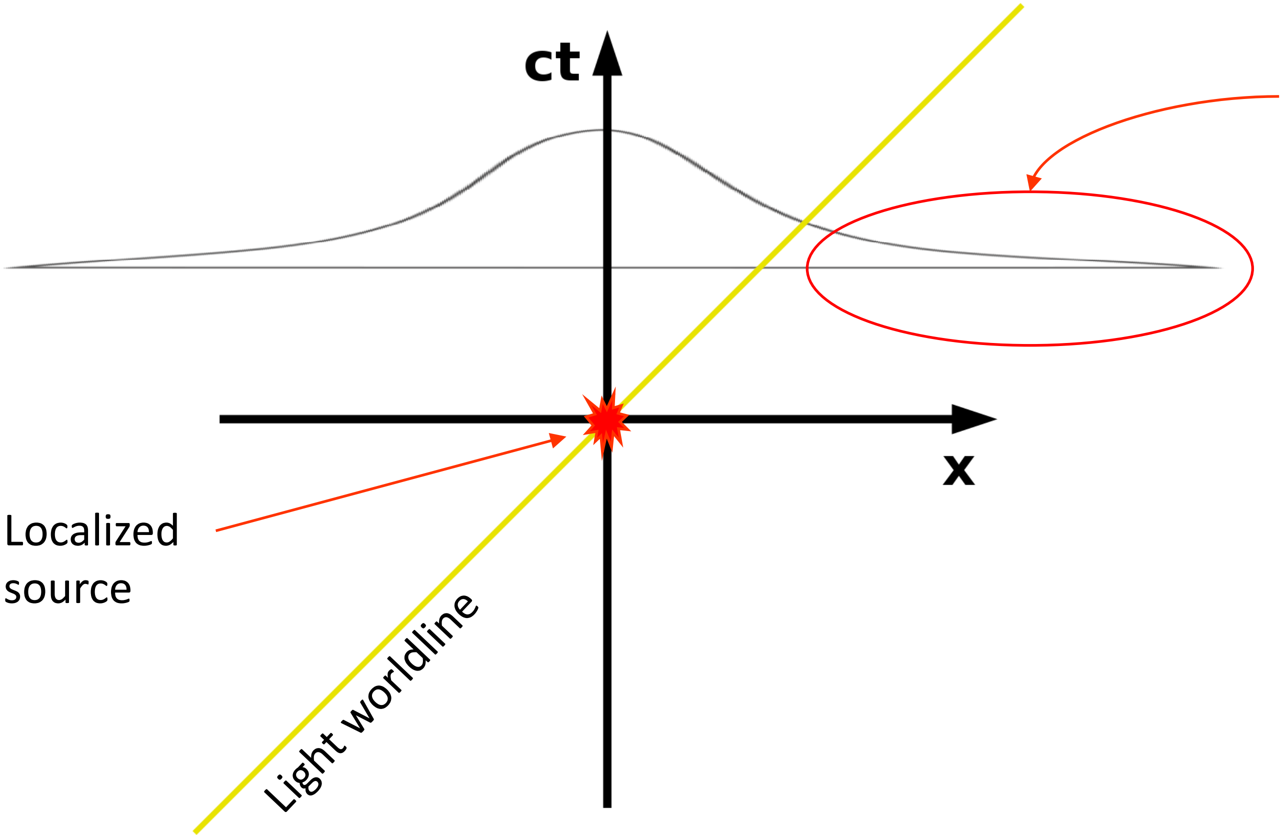
$t = 1000$



Does it work in relativity?

The Green function is $G(x,t) = \frac{1}{\sqrt{4\pi D t}} exp\left(-\frac{x^2}{4Dt}\right)$

It describes how an initial condition $\delta(x)$ evolves in time

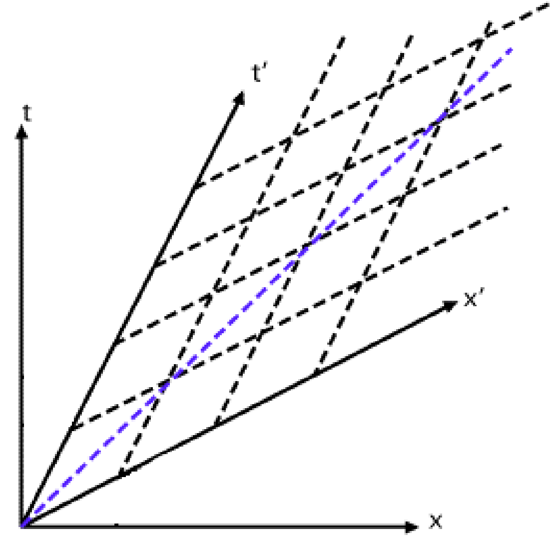


Faster than light communication.

Causality broken!

What if we "Boost" it?

If we apply the Lorentz transformation: $\begin{cases} t' = \gamma(t - vx) \\ x' = \gamma(x - vt) \end{cases}$



$$\frac{\partial T}{\partial t} = D\,\frac{\partial^2 T}{\partial x^2}$$ becomes

$$\frac{\partial T}{\partial t'} - v\,\frac{\partial T}{\partial x'} = D\gamma\left(\frac{\partial^2 T}{\partial x'^2} - 2v\,\frac{\partial^2 T}{\partial x' \partial t'} + v^2 \boxed{\frac{\partial^2 T}{\partial t'^2}}\right)$$

A second-order term in time

Warning! $T^{-1} := \sqrt{-\beta_\nu \beta^\nu}$ so I do not need to "transform" T

The state-space in the boosted frame is larger! There are more degrees of freedom:

In the frame of the medium (T) In the boosted frame $\left(T, \frac{\partial T}{\partial t'}\right)$

Instability

Let us study the homogeneous solutions in the boosted frame

$$\frac{\partial T}{\partial t'} - v\frac{\partial T}{\partial x'} = D\gamma\left(\frac{\partial^2 T}{\partial x'^2} - 2v\frac{\partial^2 T}{\partial x' \partial t'} + v^2\frac{\partial^2 T}{\partial t'^2}\right)$$

Instability

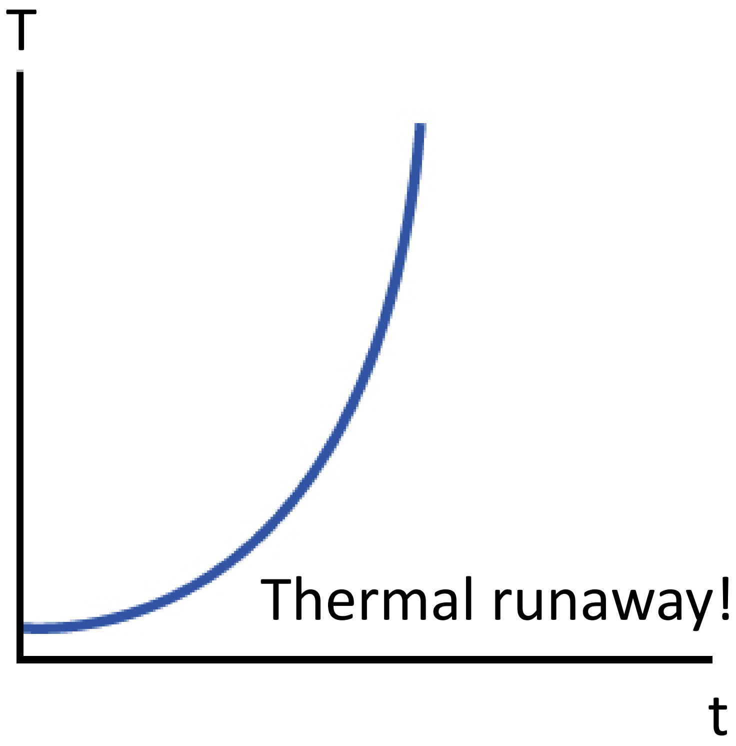
Let us study the homogeneous solutions in the boosted frame

$$\frac{\partial T}{\partial t'} - v\cancel{\frac{\partial T}{\partial x'}} = D\gamma\left(\cancel{\frac{\partial^2 T}{\partial x'^2}} - 2v\cancel{\frac{\partial^2 T}{\partial x' \partial t'}} + v^2\frac{\partial^2 T}{\partial t'^2}\right)$$

$$\frac{\partial T}{\partial t'} = D\gamma v^2 \frac{\partial^2 T}{\partial t'^2}$$

2 parameters to set in the initial conditions instead of 1

$$T = T_0 + \frac{\dot{T}_0}{\Gamma_+}\left(e^{\Gamma_+ t'} - 1\right) \qquad \Gamma_+ = \frac{1}{D\gamma v^2} > 0$$



Our freedom of setting $\dot{T}_0 \neq 0$ creates a class of solutions which explode for $t' \to +\infty$

This instability has no Newtonian analogue.

If I go back to the rest-frame of the medium

$$T(x,t) \sim e^{\Gamma_+ \gamma (t - vx)}$$

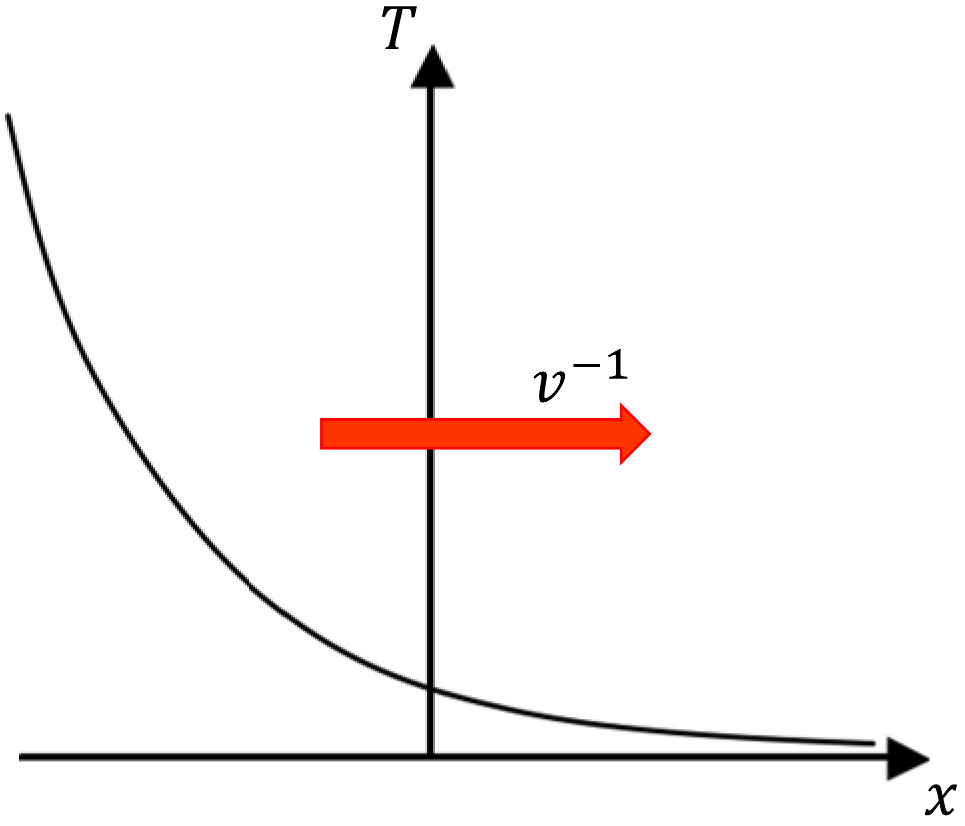
Initial thermal profile: $T(x,0) \sim e^{-\Gamma_+ \gamma v x}$

Space-time dependence of the kind: $x - \dfrac{1}{v} t$

Exponential profile which shifts rigidly faster than light!
Completely non-realistic situation:

1) Strong acausality
2) Infinite temperature for $x \to -\infty$
3) Incompatible with the assumptions which lead to the diffusion equation in the first place

Kostaedt & Liu (2000): 10.1103/PhysRevD.62.023003



This instability is unphysical, but, working in the boosted frame, we would need to fine-tune the initial conditions to avoid it.

The Eckart approach to dissipation

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \Pi g^{\mu\nu} + \Pi^{\mu\nu}$$

Heat flux　　Bulk-viscous stress　　Shear-viscous stress

In Newtonian physics:

Fourier Law: $\boldsymbol{q} = -k\nabla T$

Navier-Stokes: $\Pi = -\zeta \partial_j u^j$

$$\Pi^{jk} = -\eta\left(\partial^j u^k + \partial^k u^j - \frac{2}{3}\partial_l u^l \delta^{jk}\right)$$

All the dissipative pieces in the stress-energy tensor are assumed proportional to spatial gradients

The Eckart approach to dissipation

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \Pi g^{\mu\nu} + \Pi^{\mu\nu}$$

Heat flux | Bulk-viscous stress | Shear-viscous stress

In Newtonian physics:

Fourier Law: $\boldsymbol{q} = -k\nabla T - \mathrm{k}\mathrm{T}\mathbf{a}$

Navier-Stokes: $\Pi = -\zeta \partial_j u^j$

$$\Pi^{jk} = -\eta\left(\partial^j u^k + \partial^k u^j - \frac{2}{3}\partial_l u^l \delta^{jk}\right)$$

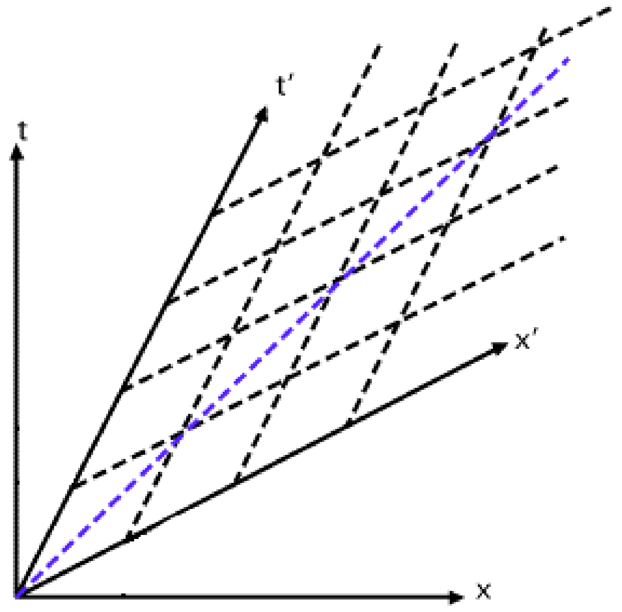
All the dissipative pieces in the stress-energy tensor are assumed proportional to spatial gradients

Eckart in essence: almost all the Newtonian relations still hold… in the reference frame of the fluid element.

Again… a second derivative!

If the fluid element is moving the derivatives in space are boosted:

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - v\frac{\partial}{\partial t'}\right)$$



This produces derivatives in time with no Newtonian analogue.
The dissipative pieces, thus, acquire time-derivative terms, e.g.

$$\Pi = -\zeta\partial_j u^j \xrightarrow{\text{Boost}} \Pi = -\zeta\partial_\nu u^\nu = -\zeta\left(\partial_j u^j + \partial_t u^t\right)$$

On the other hand, the equations of motion are simply the energy-momentum and particle conservations which, in turn, involve an other derivative in time

$$\partial_\mu T^{\mu\nu} = \partial_j T^{j\nu} + \partial_t T^{t\nu} = 0$$

Same as the heat equation: the Navier-Stokes equations, which were first-order in time in Newtonian physics, become second order in relativity!

Again…

Linear stability: we look for solutions of the form

$$f(x,t) = f_0 + \delta f\ e^{i(kx-\omega t)}$$

Equilibrium solution Small perturbation

We obtain a collection of dispersion relations

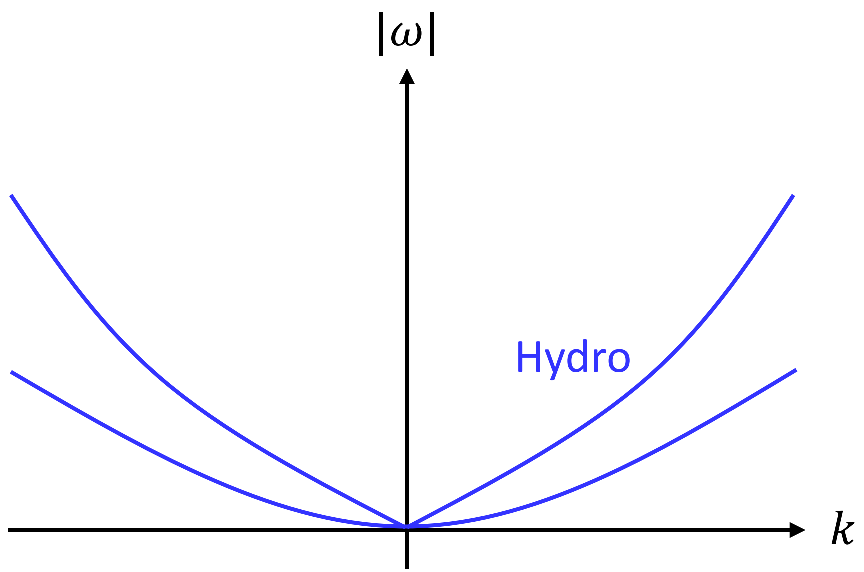
$$\omega_j = \omega_j(k)$$

In Newtonian Navier-Stokes one only finds the Hydro-modes:

Sound waves $\omega_{SO} = \pm c_s k\ - i\ \Gamma_{SO} k^2 + O(k^3)$ $\Gamma_{SO} > 0$

Shear waves $\omega_{SH} = -i\ \Gamma_{SH} k^2 + O(k^3)$ $\Gamma_{SH} > 0$

They are gapless and stable.



Gapless: $\lim_{k\to 0} \omega(k) = 0$

Stable: $Im(\omega) \leq 0$

… *an explosion!*

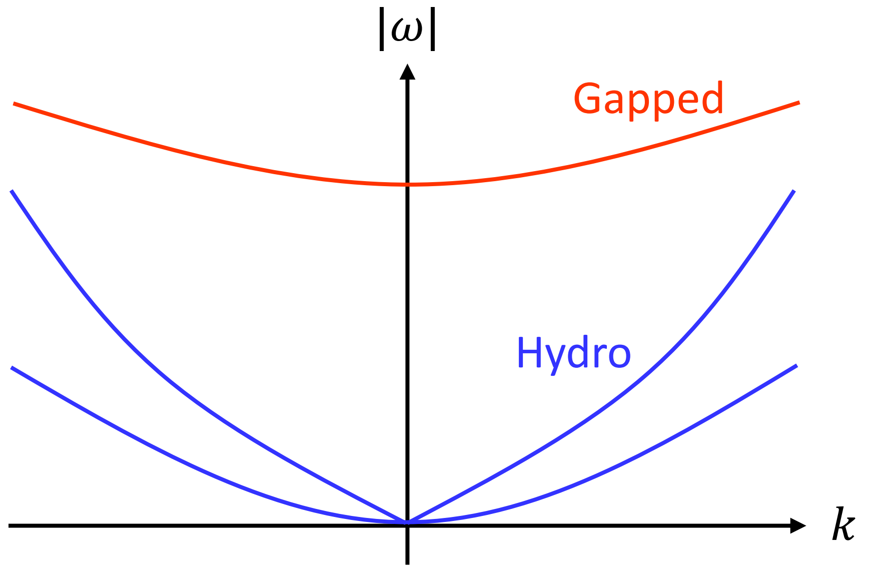
Linear stability: we look for solutions of the form

$$f(x,t) = f_0 + \delta f\ e^{i(kx-\omega t)}$$

Equilibrium solution　　　　Small perturbation

We obtain a collection of dispersion relations

$$\omega_j = \omega_j(k)$$



In Eckart theory one still finds the Hydro-modes

Sound waves　$\omega_{SO} = \pm c_s k\ - i\ \Gamma_{SO} k^2 + O(k^3)$　$\Gamma_{SO} > 0$

Shear waves　$\omega_{SH} = -i\ \Gamma_{SH} k^2 + O(k^3)$　$\Gamma_{SH} > 0$

Gapped: $\lim_{k\to 0} \omega(k) \neq 0$

Unstable: $Im(\omega) > 0$

But also some gapped mode (some mode which survives in the homogeneous limit)

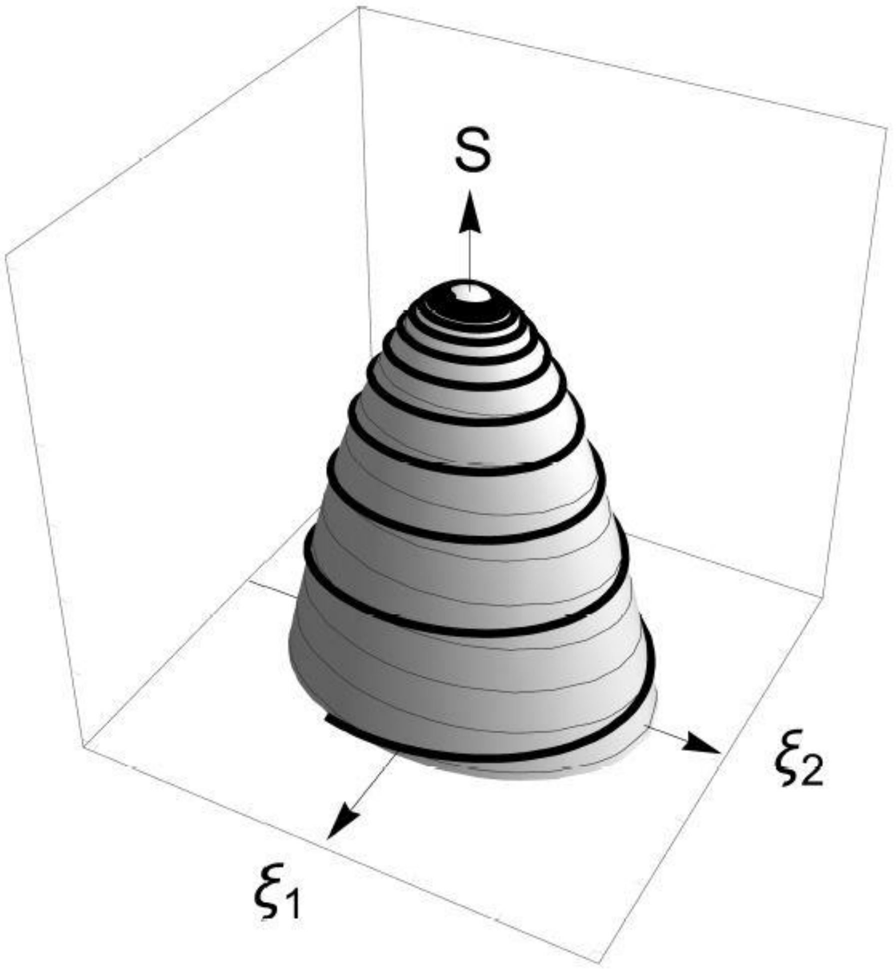
$\omega_G = i\ \Gamma_G + O(k^2)$　　$\Gamma_G > 0$　　which turns out to be unstable!

The gapped mode exists because of the higher order in time of the equations.

What went wrong?

- Every thermodynamic system admits a maximum entropy state.
- Since the entropy grows, the system will eventually converge to this state for every initial condition.
- This state is the thermodynamic equilibrium (which is necessarily stable under perturbations: Lyapunov criterion).

A system in thermodynamic equilibrium can never exhibit instabilities



The origin of the problems

Newtonian Navier-Stokes:

The only degrees of freedom are the thermodynamic fields

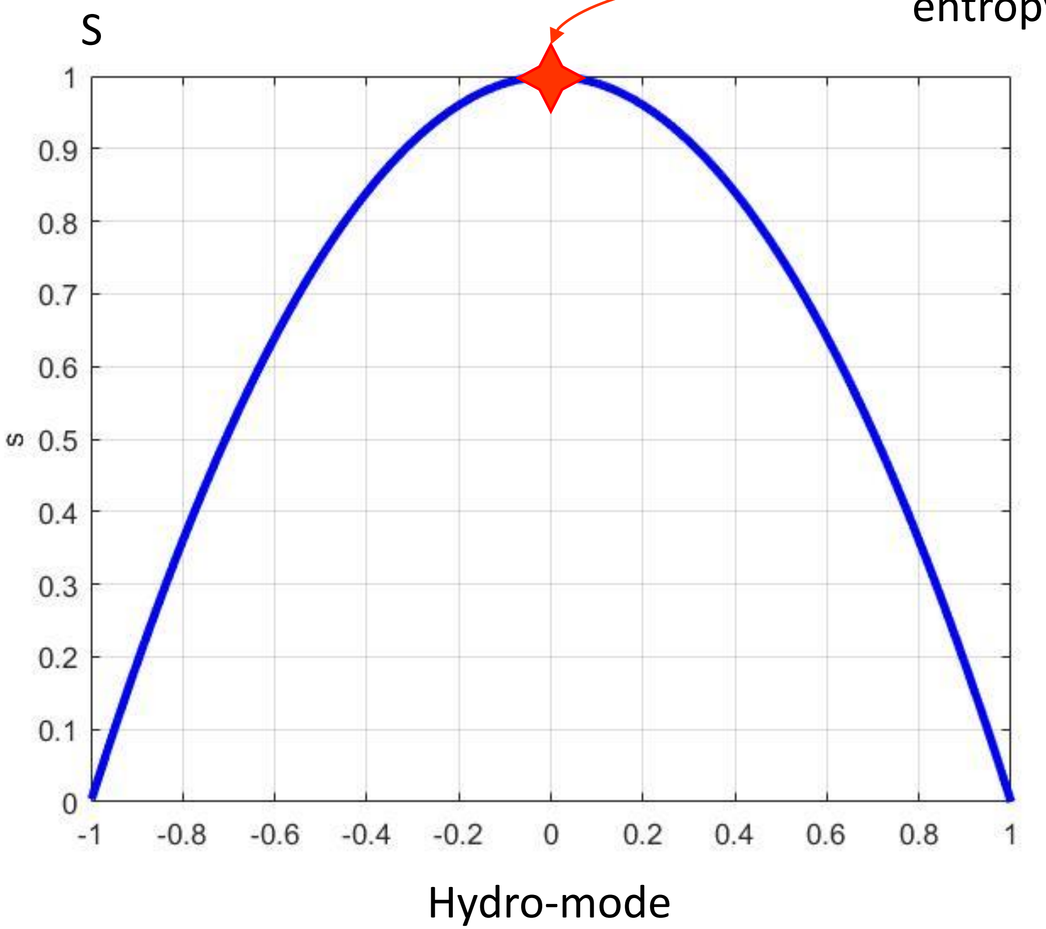
$$T, \mu, u^j$$

The number of constants of motion equals the number of thermodynamic fields

$$E, N, p^j$$

As a consequence, for fixed constants of motion, there is only one homogeneous state. Gapped modes cannot exist.

Equilibrium = Maximum entropy state



The total entropy is always maximised in homogeneous states (if basic thermodynamic conditions are respected). Therefore the Hydro-modes necessarily reduce the entropy.

In conclusion:
the equilibrium is stable in Newtonian Navier-Stokes

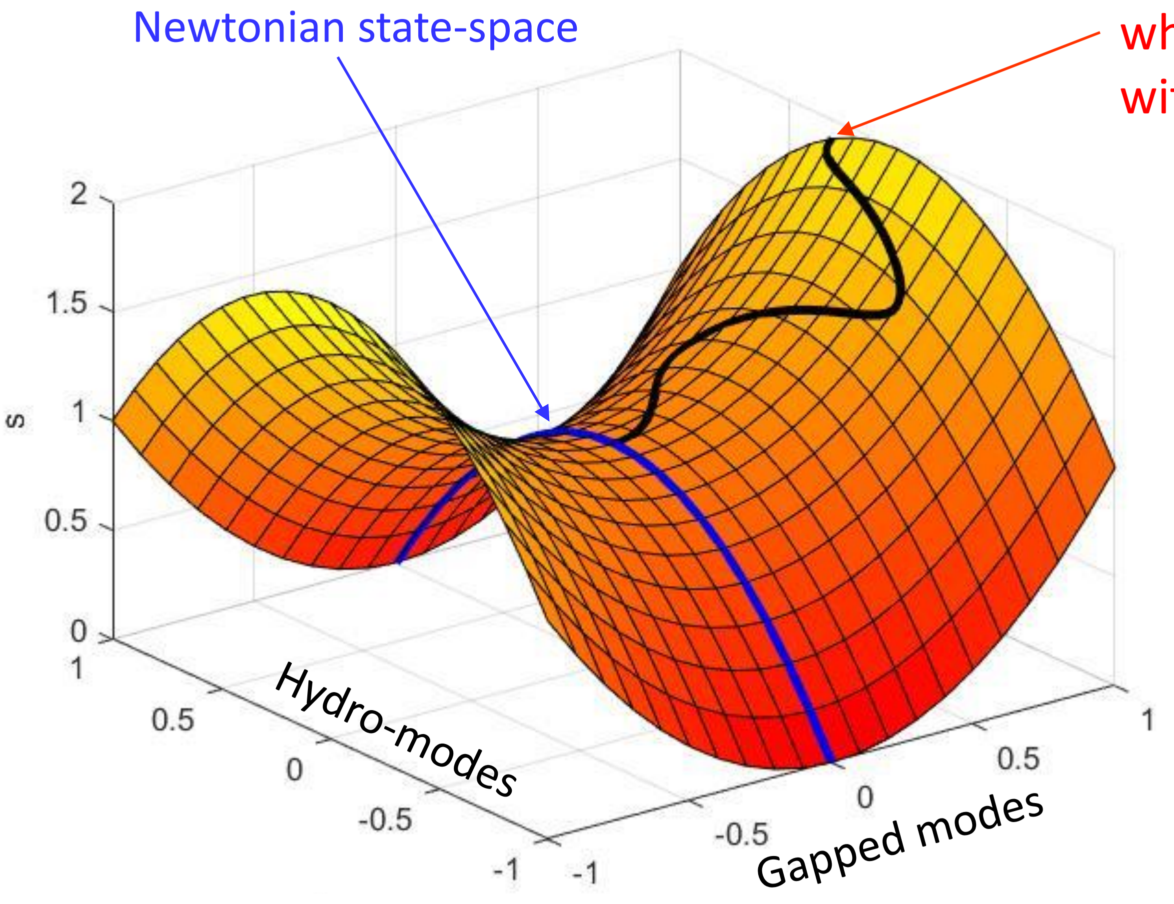
The origin of the problems

Eckart theory:

The degrees of freedom now are the thermodynamic fields and their derivatives in time

$$T, \mu, u^j, \partial_t T, \partial_t \mu, \partial_t u^j$$

Their number exceeds the number of constants of motion. Therefore there is room for a large variety of new homogeneous configurations which are dynamically accessible. Gapped modes are a necessity.



It happens that in the Eckart theory the entropy grows along the gapped modes.

The instability is thermodynamical! The obedience of the system to the second law is the very origin of the runaway!

Should hydrodynamics describe the gapped modes?

No!

"The gapped mode which is responsible for the instability is outside of the validity regime of the hydrodynamic approximation"*

- Frame-Stabilised first-order theories (*Bemfica-Disconzi-Noronha-Kovtun*)

*P. Kovtun: [10.1007/JHEP10(2019)034](https://doi.org/10.1007/JHEP10(2019)034)

Yes!

"In any realistic physical theory we expect that if a thermodynamic force is switched on/off a *relaxation time* will lapse before the corresponding thermodynamic flux is switched on/off"*

- Second-order theories (*Israel-Stewart*)
- Divergence-type theories (*Liu-Muller-Ruggeri*)
- Variational approach (*Carter*)

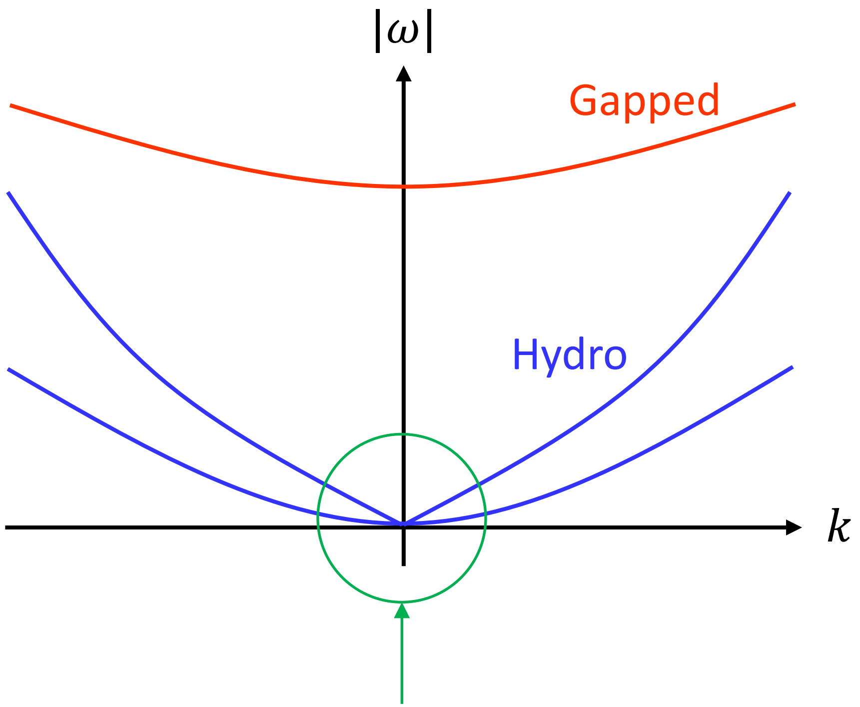
*L. Rezzolla, O. Zanotti: Relativistic hydrodynamics
Oxford University press

The way of the No: in pills

Do you need to fine-tune the initial conditions?
Just let the equations do it for you!

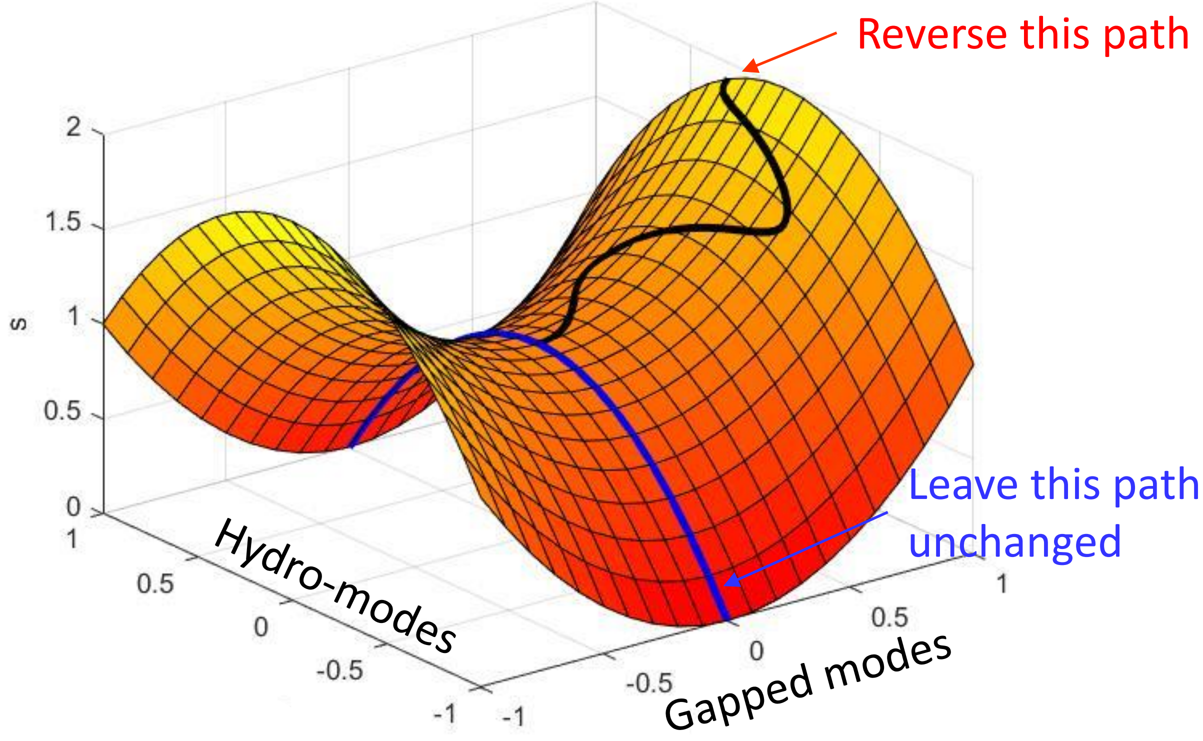
The way of the No: kill the gapped modes fast

Hydrodynamics is fundamentally an infrared theory:
It works only in the limit $\omega, k \to 0$



Only this region can be reliably described using a hydro approach, you should not trust the rest

There is no point to improve the description of the gapped modes



Just make sure that they die fast!

The way of the No: In practice

Perform a first-order derivative expansion of the physical tensors

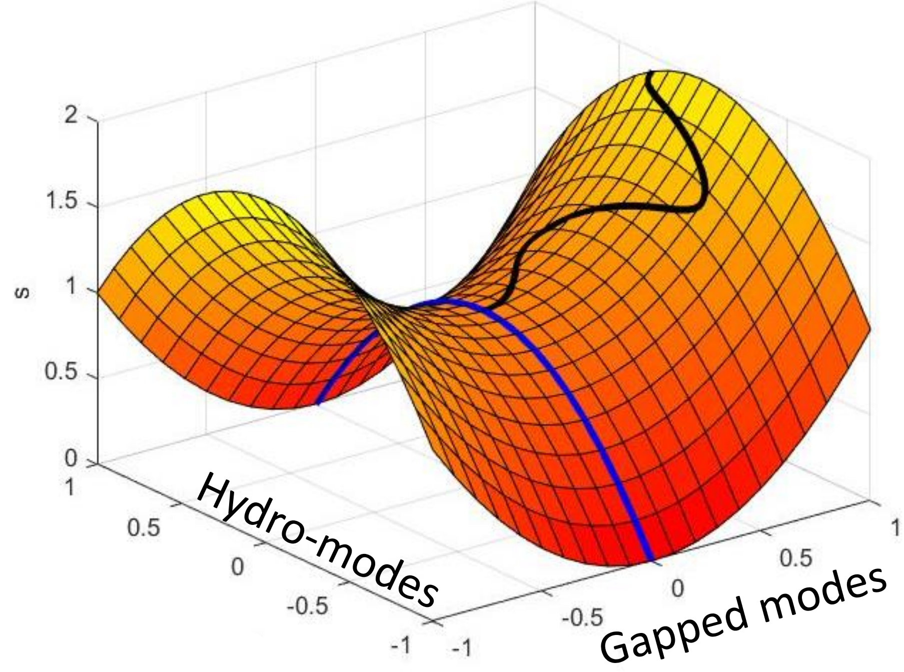
$$T^{\mu\nu} = O(1) + O(\partial)$$

Perfect fluid part

Viscous part

You include every possible contribution to $O(\partial)$. Thus there is a large number of free parameters to fix. In total 16.
How do you fix them? 3 steps:

1) Along the hydro modes (in the IR limit) every theory is equivalent to Eckart (or Landau-Lifshitz, if you prefer). Fix the parameters to reproduce the viscosity and conductivity coefficients that you want.
2) Use the remaining freedom that you have to ensure the stability of the Gapped modes. Break the second law along them.
3) There is still enough freedom to make your theory causal!



Change of frame: two first-order theories are connected by a change of “hydrodynamic frame” if they have the same behaviour along the hydro modes, but different behaviour along the gapped modes.

The way of the Yes: in pills

Do the gapped modes survive in the homogeneous limit?

Then they are thermo-modes!

Extended irreversible thermodynamics

In the microscopic theory the components of the stress-energy tensor are independent degrees of freedom.
Only in the slow limit Navier-Stokes-Fourier relations hold… the gapped modes are not slow!

Intuitively:

$$\tau\dot{\Pi} + \Pi = -\zeta\nabla_\nu u^\nu \xrightarrow{\text{Along hydro-modes}} \Pi = -\zeta\nabla_\nu u^\nu$$

Navier-Stokes hydrodynamics

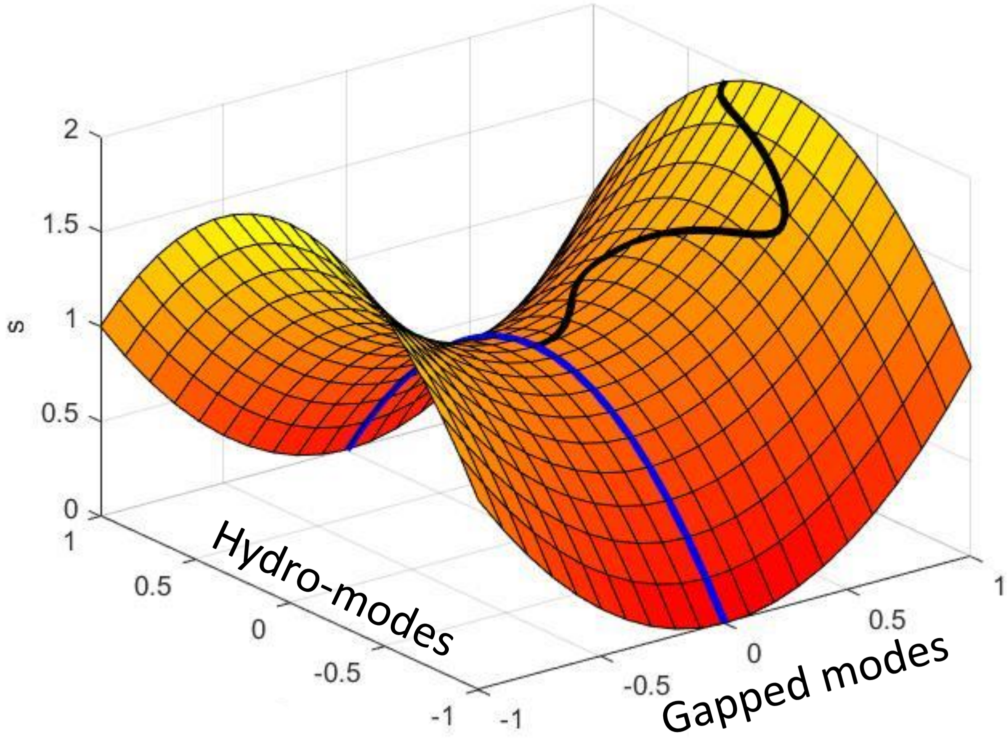
$$\tau\dot{\Pi} + \Pi = -\zeta\nabla_\nu u^\nu \xrightarrow{\text{Along gapped modes}} \tau\dot{\Pi} + \Pi = 0$$

Non-equilibrium thermodynamics

The way of the Yes aims to merge hydrodynamics and non-equilibrium thermodynamics

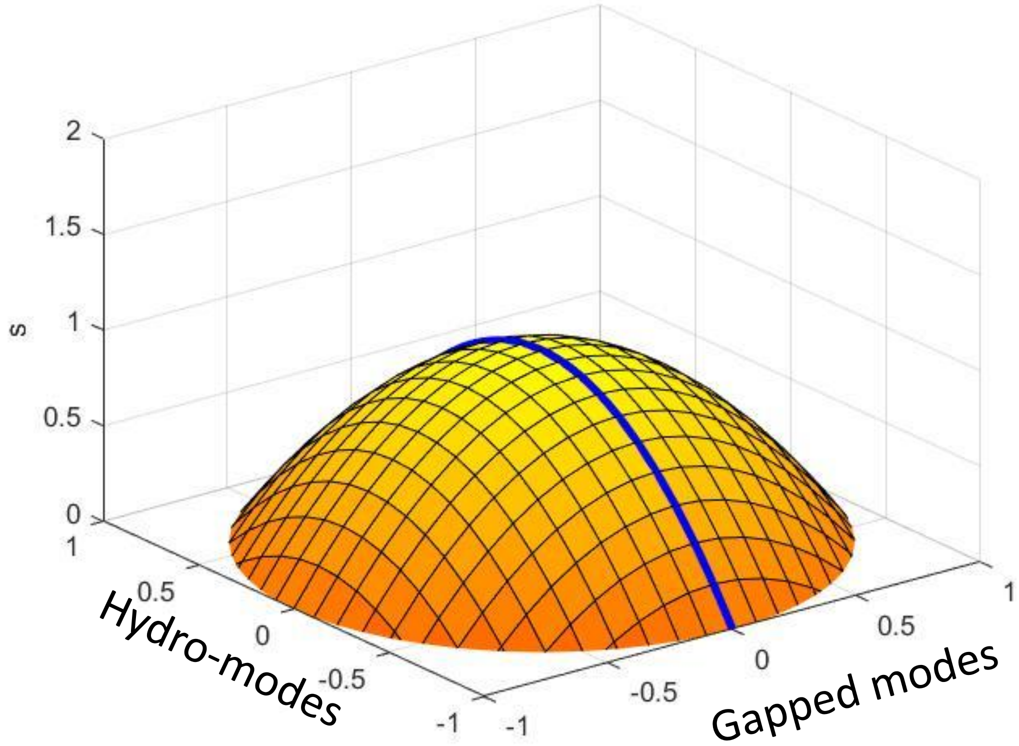
The way of the Yes: fix the entropy

Clearly, the entropy is not realistic,
it should always admit a maximum



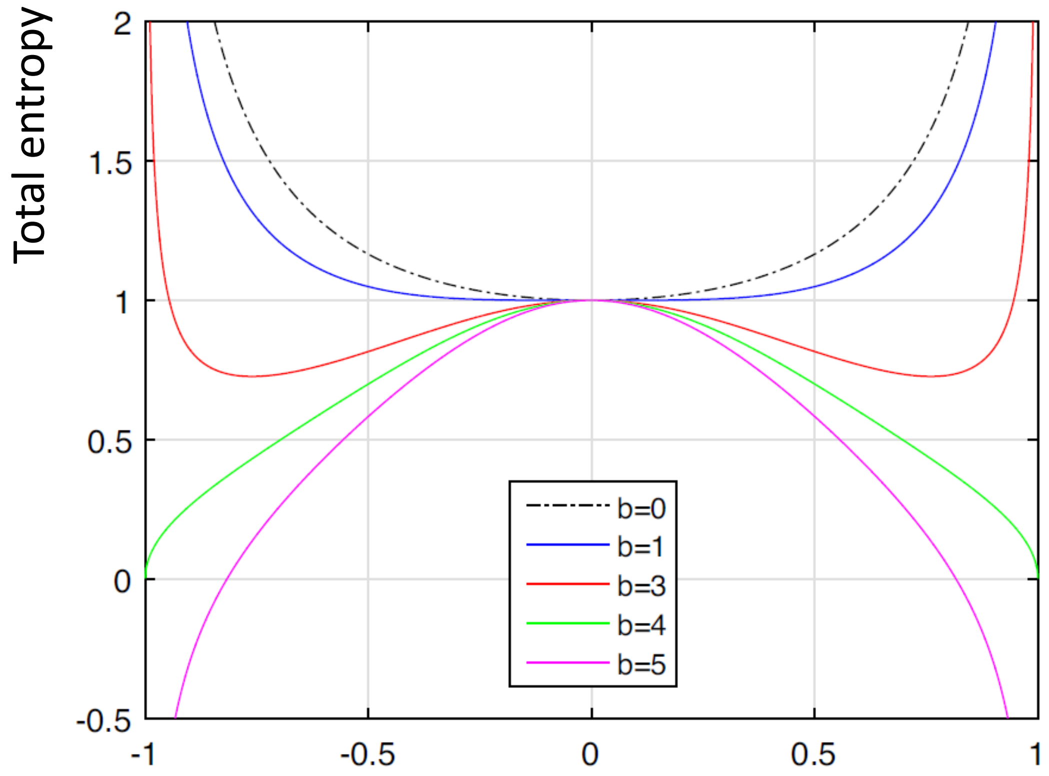
But to study the Hessian of the entropy ALL the second-order contributions are needed

We need to go to the second order also in the entropy current!



Our result: the stability conditions for Israel-Stewart are those which make the entropy maximum in equilibrium

The essence of our paper



Amplitude of the gapped mode $k = 0$

Israel-Stewart entropy current

$$s^{\nu} = su^{\nu} + \frac{q^{\nu}}{T} - \frac{b\, q^{\alpha} q_{\alpha}}{2T(\rho + P)} u^{\nu}$$

Eckart first-order part

Second-order correction

Hiscock & Lindblom (1983) stability condition

$$b > 1$$

Yes or No?

First of all you should answer this question: “is your system really in the infrared limit?”

If yes, then the choice should be based on

1) Practical convenience (e.g. which one is easier to employ in numerical simulations?)
2) The importance of the entropy in your study (e.g. do you need to have an exact Lyapunov function?)
3) Mathematical robustness (e.g. do you want to ensure strong hyperbolicity?)

Note that along the hydro-modes, in the Infrared limit, the two theories are equivalent, thus no microscopic argument can be used to select a “better” theory.

If the system is not in the infrared limit, then the first-order theories are inapplicable by construction.
In this case one should rely on a second-order approach (or abandon hydrodynamics altogether).

…*But Neutron Stars are NOT Infrared!*

If you make microscopic calculations for the damping NS oscillations (hydro-modes) you find:

$$Im(\omega) = -\frac{1}{2}\frac{k^2}{\rho + P}\frac{A^2}{1 + B^2 Re(\omega)^2}$$

…But Neutron Stars are NOT Infrared!

If you make microscopic calculations for the damping NS oscillations (hydro-modes) you find:

$$Im(\omega) = -\frac{1}{2}\frac{k^2}{\rho + P}\frac{A^2}{1 + B^2 Re(\omega)^2}$$

Assuming Navier-Stokes hydrodynamics you can derive an effective bulk viscosity making the identification

$$Im(\omega) = -\frac{1}{2}\frac{\zeta\, k^2}{\rho + P} \longrightarrow \zeta = \frac{A^2}{1 + B^2 Re(\omega)^2}$$

The bulk viscosity depends on the frequency!
This effect is usually non-negligible.

…But Neutron Stars are NOT Infrared!

If you make microscopic calculations for the damping of NS oscillations (hydro-modes) you find:

$$Im(\omega) = -\frac{1}{2}\frac{k^2}{\rho + P}\frac{A^2}{1 + B^2 Re(\omega)^2}$$

Assuming Navier-Stokes hydrodynamics you can derive an effective bulk viscosity making the identification

$$Im(\omega) = -\frac{1}{2}\frac{\zeta\, k^2}{\rho + P} \quad \longrightarrow \quad \zeta = \frac{A^2}{1 + B^2 Re(\omega)^2}$$

The bulk viscosity depends on the frequency!

If, instead, you assume Israel-Stewart hydrodynamics, then you need to impose

$$Im(\omega) = -\frac{1}{2}\frac{k^2}{\rho + P}\frac{\zeta}{1 + \tau^2 Re(\omega)^2} \quad \longrightarrow \quad \zeta = A^2 \qquad \tau = B^2$$

Neutron-star matter is the prototype of an Israel-Stewart fluid, Extended Irreversible Thermodynamics is necessary!

In summary

1) The relativity of simultaneity creates new degrees of freedom with no Newtonian analogue.
2) In this new extended state-space the entropy is no longer maximised in equilibrium.
3) The strict obedience of the system to the second law is the real source of instability.
4) To solve this issue, either you break the second law, or you fix your entropy.

In summary

1) The relativity of simultaneity creates new degrees of freedom with no Newtonian analogue.
2) In this new extended state-space the entropy is no longer maximised in equilibrium.
3) The strict obedience of the system to the second law is the real source of instability.
4) To solve this issue, either you break the second law, or you fix your entropy.

Thank you for your attention!

Appendices

The Cattaneo solution

Fourier law

$$q = -k\nabla T$$

Diffusion equation

$$\frac{\partial T}{\partial t} = D\,\frac{\partial^2 T}{\partial\, x^2}$$

Cattaneo law

$$\tau\dot{q} + q = -k\nabla T$$

Relaxation term, introduces a finite signal-propagation speed

Telegraph equation

$$\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = D\,\frac{\partial^2 T}{\partial\, x^2}$$

Now the equation is of the second order also in the rest-frame

It was designed to fix the causality problem, but it turned out to fix also the stability.
Our aim is to prove rigorously the universality of this idea for every dissipative process.

A glimpse into how it works

Example:

Mixture of particles of type p and n (the two species comove)

Equation of state: $s = s(\rho, n_p, n_n)$

Assume there is a reaction of the type

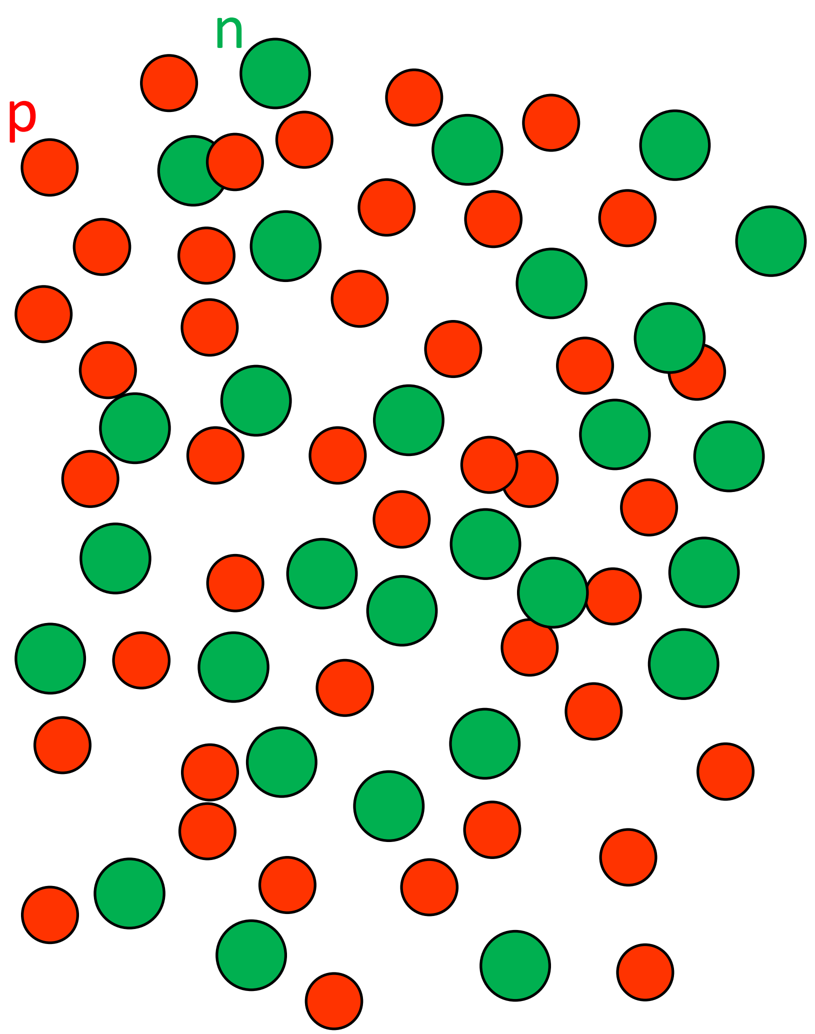
$$p \leftrightharpoons n$$

The chemical evolution is governed by the equation

$$\nabla_\sigma n_p^\sigma = -\nabla_\sigma n_n^\sigma = \Xi \mathrm{A} \qquad \mathrm{A} = \mu_n - \mu_p \qquad \Xi > 0$$

It is a dissipative process (it produces entropy)

$$T \nabla_\sigma s^\sigma = \Xi \mathrm{A}^2 \geq 0$$



No magic required…

Some notation:

Volume per particle: $v = \left(n_p + n_n\right)^{-1}$

Entropy per particle: $x_s = vs$

Type-p particle fraction: $x_p = vn_p$

Conglomerate four-velocity: $u^\sigma = \frac{n_p^\sigma}{n_p} = \frac{n_n^\sigma}{n_n}$

Continuity equation: $\dot{v} = v\nabla_\sigma u^\sigma$

Chemical evolution: $\nabla_\sigma n_p^\sigma = \Xi \mathrm{A}$

Evolution of the type-p particle fraction: $\dot{x}_p = v\Xi \mathrm{A}$

Change of variables: $x_p = x_p(v, x_s, A)$

$$\frac{\partial x_p}{\partial v}\dot{v} + \cancel{\frac{\partial x_p}{\partial x_s}\dot{x}_s} + \frac{\partial x_p}{\partial A}\dot{A} = v\Xi \mathrm{A}$$

Neglect second order in the affinity

Introduce new coefficients:

$$\xi = \frac{1}{\Xi}\frac{\partial x_p}{\partial v}$$

$$\tau = -\frac{1}{v\Xi}\frac{\partial x_p}{\partial A}$$

$$\boxed{\tau\dot{A} + A = \xi\, \nabla_\sigma u^\sigma}$$

A universal structure

Heat flux $\tau\dot{q} + q = -k\nabla T$

Chemical affinity $\tau\dot{A} + A = \xi\, \nabla_\sigma u^\sigma$

They have the same mathematical structure!

We proved that any thermodynamically consistent model for bulk viscosity has a relaxation term:

$$\tau\dot{\Pi} + \boxed{\Pi = -\zeta\, \nabla_\sigma u^\sigma}$$

Israel-Stewart relaxation term

Navier-Stokes part

Thermodynamics automatically forbids the instability, and to do so it produces these compensating terms which cannot be neglected in relativity. This always turns out to make the theory also causal.

We do not need a new theory. We need to understand the thermodynamics content of the theories we already have.

In conclusion

- Thermodynamics rules once again.

In conclusion

- Thermodynamics rules once again.

My papers of the academic year (all submitted, waiting for the referee report):

- **Bulk viscosity in relativistic fluids: from thermodynamics to hydrodynamics** arXiv:2003.04609
- **The zeroth law of thermodynamics in special relativity** arXiv:2005.06396
- **When the entropy has no maximum: a new perspective on the instability of the first-order theories of dissipation** arXiv:2006.09843

Thank you for your attention!